On the contact between two linearly elastic bodies

Thesis submitted in partial fulfillment of the requirements for the degree

Master of Science

Presented by

Miguel de Benito Delgado

under the supervision of
Prof. Dr. Jürgen Scheurle

Fakultät für Mathematik,
Technische Universität München

Munich, March 2013

On the contact between two linearly elastic bodies

Thesis submitted in partial fulfillment of the requirements for the degree
Master of Science

Presented by
Miguel de Benito Delgado

under the supervision of
Prof. Dr. Jürgen Scheurle

on the 21st of March, 2013

Fakultät für Mathematik,
Technische Universität München

ABSTRACT. [1] We first introduce the problem of the contact of two elastic bodies in the context of classical linear elasticity, then gather all necessary results for the proof of existence and uniqueness of a solution. The main proof of existence is conducted using two different approaches, depending on the coercivity of the bilinear form associated with the elastic potential. If coercivity is present, the answer is simple with a few tools from convex analysis, which we develop in §A, but in the opposite case a lengthier approach is required, which we base on the classical paper [26]. We collect these results in §3.2. Among the technical requisites for a general and rigorous formulation in the weak sense, the trace space $H_{00}^{1/2}$ of J.L. Lions and E. Magenes stands out for its subtle role in Green's formula, an issue we deal with in §B. Finally, in §C we discuss Korn's inequality and the role it plays, and provide a proof based on (a generalization of) a celebrated lemma by Lions.

In section 4 we briefly present a finite element discretization and an algorithm for the solution. In particular we focus on some fairly recent advances in the resolution of the non-linearity at the contact zone via iterative methods, a strategy put forth in [18], where a mortar method with dual Lagrange multipliers [33] is used to couple the bodies at the contact boundary and the contact set is determined by alternating nodes between Dirichlet and Neumann conditions. As an illustration, we apply these techniques to the simple case of two elastic plates on the plane.

1. This document was entirely created using the open source scientific platform $\mathrm{T_EX_{MACS}}$: www.texmacs.org

Ich erkläre hiermit, dass ich diese Master's Thesis selbständig und nur mit den angegebenen Hilfsmitteln angefertigt habe.

München, 21. März 2013

Table of contents

1. Introduction

When two deformable bodies are pressed together forces appear at the surface where they meet. In the simplest model they are of two kinds: the forces in the direction normal to the deformation account for the bodies not penetrating each other and those in tangential directions for the friction. The study of the deformations and stresses occurring under volume and surface loads imposed on the bodies in the stationary case is called *contact elastostatics* and leads to free boundary problems given by a system of elliptic PDE with non-linear boundary conditions. Characteristic of contact problems are conditions of *Signorini* type, given in an unknown subset of the boundary by inequalities expressing the requisite that the bodies don't interpenetrate.[1.1]

Contact problems in elasticity have played a very important role in applications since the 19th century. These involve the study of questions ranging from the deformation of beams and plates bringing them into contact with each other or supporting structures, to the behaviour of human bones and prostheses under load, and consequently they have had an enormous industrial and scientific applicability which has motivated a vast body of research both in the engineering and mathematical literature.

The former officially starts with Hertz's paper [16] in 1881, where the point pressure between two bodies is approximated using analytical tools and explicit expressions are derived for several particular cases. Supposing axial symmetry and using approximations of low order, Hertz derives conditions on the vertical displacements involving simple functions of the principal curvatures of paraboloids. This is exploited together with the Green function for a half plane to arrive at formulas which have been in widespread use for over a century (see e.g. [30]). Later in the engineering literature, more geometrical simplifications and regularity assumptions were used in order to adapt the tools of classical mechanics, in particular the classical variational formulation, and succeeded in finding usable approximations (see [21] and the references in its §1). But however successful these approaches may be in particular cases, they remain ad-hoc and the amount of simplifications needed make them unsuitable for generalization to more complex problems.

Around the 1950s a very fruitful school of italian and french mathematicians, starting with Antonio Signorini and his student Gaetano Fichera, began the rigorous study of the questions of existence, uniqueness and regularity of solutions to the problems of contact elastostatics. Signorini first investigated the stationary equilibrium of one elastic solid over a rigid foundation and suggested the question in 1933 at the *Istituto Nazionale di Alta Matematica*. It was the ambiguity due to the unknown contact zone in the boundary that led him to question whether the problem was correctly posed in the first place. The positive answer was given by Fichera in a series of two papers, the last providing the proof of existence and uniqueness of solution in 1964.[1.2]

The rich field of variational (in)equalities quickly sprouted, with the seminal paper by Jacques-Louis Lions and Guido Stampacchia [26] as a central player and, to cite only two celebrated examples, the book by Duvaut and Lions [10] with a focus in mechanics and, already in the 1980s, the introduction to the subject by Kinderlehrer and Stampacchia [23]. The techniques developed are at the heart of many free boundary problems, whose non-linearity in the boundary conditions is translated into inequalities, the classical example being the Stefan problem for ice fusion.[1.3]

1.1. After the italian mathematician Antonio Signorini who first posed the problem of one elastic body resting on a rigid foundation in 1933. His original name for them was *ambiguous conditions*.

1.2. G. Fichera, "Problemi elastostatici con vincoli unilaterali: Il problema di Signorini con ambigue condizioni al contorno.", *Atti della Accademia Nazionale dei Lincei. Memorie. Classe di Scienze Fisiche, Matematiche e Naturali*, **7** (1964), pp. 91–140.

Related to these advances is the numerical analysis of contact problems, a tool indispensable to many areas of modern science and industry which in a sense closes the loop starting with Hertz's paper. Although his formulas for the contact of linearly elastic bodies in the shape of cylinders, spheres and planes are used even today, the great industrial interest of contact problems and the lack of explicit analytic solutions have motivated intense research for decades. The book [22] by Kikuchi and Oden is perhaps the first unified treatise on the analysis and numerical approximation of unilateral contact problems. Here the question is mainly considered from the perspective of convex minimization and the numerical schemes proposed focus on penalty methods: a non-negative term is added to the energy functional which grows greatly when the deformation violates the contact condition. To address the non-linearity that (inevitably) appears an iterative scheme is proposed. A fundamental flaw of this method is its strong sensitivity to a penalty parameter which has to be adjusted according to the parameters of the material, the mesh size, etc. This parameter greatly affects the conditioning of the system and poses problems in practical applications. An alternative approach, presented in [4], uses a technique from non-conforming domain decomposition methods, weakening the contact conditions to integrals to introduce them in the discrete system. This so called *mortar method* was modified in [33] to use dual Lagrange multipliers in a saddle point formulation thus resulting in simpler and faster implementations.

1.3. G. Duvaut, "Résolution d'un problème de Stefan (Fusion d'un bloc de glace a zero degrées)" *C.R. Acad. Sci. Paris*, **276** (1973) pp. 1461–1463.

2. The classical formulation

We first quickly review the physical model involved, assuming all functions and domains to be as smooth as necessary in order for the operations we make with them to make sense. Then we derive boundary conditions which capture the ideas of non-penetration, contact and lack of friction. Finally we write the equations in their classical form.

2.1. The physical model.

We study the problem of contact between two elastic bodies $\mathcal{B}^a$ and $\mathcal{B}^b$ in $\mathbb{R}^n$.[2.1] In its reference configuration (i.e. in the absence of any force) each body $\mathcal{B}^\iota$ occupies a domain $\Omega^\iota \Subset \mathbb{R}^n$, which is an open set bounded by a sufficiently smooth compact manifold $\Gamma := \partial\Omega$. **Displacements** from this configuration are given by the applications $\Omega^\iota \ni x \mapsto y = y^\iota(x)$ and **deformations** by $u^\iota(x) = y^\iota(x) - x$. The maps y^ι are assumed injective with $\det Dy^\iota > 0$.

We assume the bodies to be **homogeneous** (and with reference density 1), **anisotropic** and **hyperelastic**. In particular we shall assume *small deformations* (e.g. $|u| \ll 1$, $|\nabla u| \ll 1$ if $\operatorname{diam} \Omega^\iota \simeq 1$) and the existence of an **elastic potential**

$$e^\iota(x; u^\iota) = \frac{1}{2}\, a^\iota_{ijkl}(x)\, \varepsilon_{kl}(u^\iota)\, \varepsilon_{ij}(u^\iota),$$

where

$$\varepsilon_{kl}(u^\iota) := \frac{1}{2}\, (u^\iota_{k,l} + u^\iota_{l,k})$$

denotes as is customary the **linearized strain tensor** and the a^ι_{ijkl} are the coefficients of the Hookean elasticity tensors for each of the bodies, satisfying the following conditions:

i. $a^\iota_{ijkl} \in L^\infty(\Omega^\iota)$. In particular there exist constants $M^\iota > 0$ such that

$$\max_{1 \leqslant i,j,k,l \leqslant n} \|a^\iota_{ijkl}\|_\infty \leqslant M^\iota.$$

ii.

$$a^\iota_{ijkl} = a^\iota_{jilk} = a^\iota_{klij}. \tag{2.1}$$

iii. There exist constants $m^\iota > 0$ such that

$$a^\iota_{ijkl}(x)\, \xi_{ij}\, \xi_{kl} \geqslant m^\iota\, \xi_{ij}\, \xi_{kl} \tag{2.2}$$

a.e. in Ω^ι and for every symmetric $n \times n$ matrix $\xi = (\xi_{ij})$.

We explain with an intuitive argument that the symmetries (2.1) suppose no restriction. In $\mathbb{R}^3$ for instance they imply that a has only 21 different coefficients instead of $3^4 = 81$, but notice that no more are necessary: since ε is symmetric and has only 6 coefficients and a quadratic form in $\mathbb{R}^6$ is characterized by 21 coefficients, this is the number of a_{ijkl} that are needed.

The **strong ellipticity** condition (2.2) is a ubiquitous inequality constraint on the elasticity tensor, which not only is in many situations physically sound, but also proves to be essential to the question of existence.[2.2]

2.1. See the notation in §D.

As usual σ^ι stands for the **Cauchy stress tensor**. Balance of linear momentum for each body then reads

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega'} u^\iota_{,t}\,\mathrm{d}x = \int_{\Omega'} f^\iota\,\mathrm{d}x + \int_{\partial\Omega'} \sigma^\iota\,\nu^\iota\,\mathrm{d}s_x,$$

for any open set $\Omega' \Subset \Omega^\iota$. Differentiating under the integral sign and using Gauß' theorem we have

$$\int_{\Omega'} u^\iota_{,tt}\,\mathrm{d}x = \int_{\Omega'} f^\iota\,\mathrm{d}x + \int_{\partial\Omega'} \sigma^\iota\,\nu^\iota\,\mathrm{d}s_x = \int_{\Omega'} f^\iota + \operatorname{div}\sigma^\iota\,\mathrm{d}x,$$

and gathering terms together:

$$\int_{\Omega'} (u^\iota_{,tt} - \operatorname{div}\sigma^\iota - f^\iota)\,\mathrm{d}x = 0.$$

Because $\Omega' \Subset \Omega^\iota$ was arbitrary we arrive at

$$u^\iota_{,tt} - \operatorname{div}\sigma^\iota = f^\iota. \tag{2.3}$$

and after discarding the term related to time, and adding the tractions at the boundary we have the equations of equilibrium:

$$\begin{cases} -\operatorname{div}\sigma^\iota(u^\iota) &=& f^\iota & \text{in } \Omega^\iota, \\ \sigma^\iota(u^\iota)\cdot\nu^\iota &=& t^\iota & \text{on } \Gamma^\iota_N, \end{cases}$$

where f^ι are body forces on Ω^ι and t^ι tractions on the open subsets $\Gamma^\iota_N \subset \Gamma^\iota$. We will assume both to be **dead loads**, i.e. independent of the actual deformation of the body.

Elastic behavior of the materials means that it is assumed that deformations disappear once the forces that caused them do. This is modelled with the elastic potential introduced above and is used for the derivation of a *constitutive law* of the material yielding an expression for the Cauchy stress tensor. Starting from the equation of conservation of energy

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega'} \left(\frac{1}{2}|u^\iota_{,t}|^2 + e^\iota\right)\mathrm{d}x = \int_{\Omega'} f^\iota\cdot u^\iota_{,t}\,\mathrm{d}x + \int_{\partial\Omega'} \sigma^\iota\,\nu^\iota\cdot u^\iota_{,t}\,\mathrm{d}s_x,$$

we derive under the integral sign and use Gauß' theorem to arrive after some computations at

$$\int_{\Omega'} e^\iota_{,t}\,\mathrm{d}x = \int_{\Omega'} \sigma^\iota_{ij}\,u^\iota_{j,it}\,\mathrm{d}x,$$

for all open subsets $\Omega' \Subset \Omega^\iota$. We have therefore

$$e^\iota_{,t} - \sigma^\iota_{ij}\,u^\iota_{i,jt} = 0, \text{ in } \Omega^\iota$$

and although the energy density could in principle depend on t, x, u, Du, the previous equation and the freedom in the choice of f^ι imply that it is in fact only a function of x and Du^ι. To see this, we fully write the previous equation:

$$0 = e^\iota_{,t} + e^\iota_{,u_i}\,u^\iota_{i,t} + e^\iota_{,u_{i,j}}\,u^\iota_{i,jt} - \sigma^\iota_{ij}\,u^\iota_{i,jt}, \tag{2.4}$$

2.2. In particular it has the Baker-Ericksen inequality as a consequence, which states the intuitive principle that "stress grows with strain", see [28, §1.5].

then verify that first: e^ι *does not depend on* t, since taking $f^\iota = 0$ any constant function u^ι is a solution of (2.3) and then all terms in (2.4) but the first are zero, i.e. $0 = e^\iota_{,t}$; and second: e^ι *does not depend on* u^ι, for taking again $f^\iota = 0$, the functions $u^\iota = (0, \ldots, t, \ldots 0)$ where only the k-th component is not zero are solutions and then all terms in (2.4) disappear, except for $e_{,u_k} u_{k,t} = e_{,u_k} = 0$ for each k. We have therefore arrived at the following expression for the stress tensor:

$$\sigma^\iota_{ij}(u^\iota) = e^\iota_{,u_{i,j}}(Du^\iota) = a^\iota_{ijkl}\, \varepsilon_{kl}(u^\iota),$$

which receives the name of **generalized Hooke's law**.[2.3]

For the analysis of existence and uniqueness of solutions to our actual problem, it will be key to study not these equations but their equivalent form as a minimization problem in accordance with the principle of virtual work. The total stored energy function of each body is

$$I^\iota(u^\iota) = \int_{\Omega^\iota} e^\iota(x; u^\iota)\, \mathrm{d}x - F^\iota(u^\iota)$$

where the first term represents the work due to the stored elastic energy and the second is a linear operator representing the virtual work done by external forces. In the case these are regular enough, F will adopt the structure

$$F^\iota(u^\iota) = \int_{\Omega^\iota} f^\iota(x) \cdot u^\iota(x)\, \mathrm{d}x + \int_{\Gamma^\iota_N} t^\iota(x) \cdot u^\iota(x)\, \mathrm{d}s_x.$$

In what follows we will continue using the superscript ι to denote properties which both bodies have, and we will gather these under the same variables without superscript. Thus, the total (elastic) energy of the system will be

$$I = I^a + I^b.$$

2.2. Boundary conditions.

We now set to the task of finding suitable boundary conditions modeling the possible contact of both bodies on a subset of their boundaries and surface stresses on the rest.[2.4] As we will later see, it is the lack of knowledge of the contact zone, expressed as what Signorini named "ambiguous conditions", that is most problematic, leading to a *free boundary problem*.

First let Γ^ι_D, Γ^ι_N, Γ^ι_S be disjoint open sets where Dirichlet and Neumann data are given and contact *may occur* respectively, with Γ^ι_D and Γ^ι_N possibly empty. Because of the special role played by the essential boundary conditions, we must single out the set $\Sigma^\iota := \mathrm{Int}(\Gamma^\iota \backslash \Gamma^\iota_D)$. Under appropriate regularity assumptions on the data, we will be able to extend some of it by zero if needed thus allowing us to assume right away that $\Gamma^\iota = \overline{\Gamma}^\iota_D \cup \overline{\Gamma}^\iota_N \cup \overline{\Gamma}^\iota_S$: on those portions of Γ^ι where *a priori* there are no conditions, one can say that tractions are zero and thus in principle one can take Γ^ι_N to be $\Gamma^\iota \backslash (\overline{\Gamma}^\iota_D \cup \overline{\Gamma}^\iota_S)$ by extending t^ι by zero, but care must be had with the function spaces.[2.5] It is reasonable to say this because at a point of the boundary where there is no contact and no forces are prescribed there must appear no stress, in particular $\sigma^\iota(u^\iota)\, \nu^\iota = 0$.

2.3. We refer to [11, §5.10] and [28, Chapter 2] for further details.

2.4. The rather vague notion of *possible contact* will be made precise in the sequel.

2.5. See §B.1.

2.2.1. Kinematical non-penetration.

([2.6]) In order to derive conditions on the solution which model the requisite that the bodies do not interpenetrate each other, we assume that the actual contact zone is a smooth $n-1$ dimensional manifold Γ_C. This manifold need not be located between the boundaries *before* the deformation, but *after* it (think of two spheres touching, one much softer than the other: Γ_C will be "inside" the reference configuration of the softer one). The trick will be writing the contact condition in reference coordinates by means of a Taylor expansion.

Smoothness of the boundary manifolds and their proximity because of the assumption of small displacements, imply that they are *locally*[2.7] "almost parallel", so that there exists an open subset $U \subset \mathbb{R}^{n-1}$ and differentiable functions φ^a, φ^b, φ^c such that, possibly after a rotation, their graphs $\{(x', \varphi^\iota(x')) : x' \in U\}$ contain (locally and respectively) Γ_S^a, Γ_S^b and Γ_C. We will use $U \ni x'$ as coordinates for the boundaries. In particular we set for any function f defined on $\overline{\Omega}^\iota$:

$$f(x') := f(x', \varphi^\iota(x')) \text{ for every } x' \in U.$$

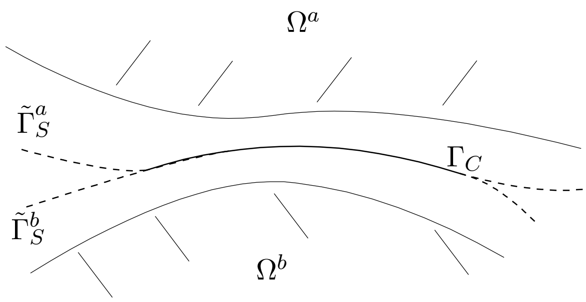


Figure 2.1. Close-up of the contact zone.

We can make more explicit this condition that the boundaries are almost parallel by stating that up to first order terms the Taylor expansions of the φ^ι are equal, which in particular means

$$\nabla \varphi^a = \nabla \varphi^b = \nabla \varphi^c \text{ in } U. \tag{2.5}$$

We will consider points in Γ_S^a and Γ_S^b as "aligned in U", i.e. $x^{a\prime} = x^{b\prime} =: x'$. The condition that Γ_C "lays between" the *deformed* surfaces $\tilde{\Gamma}_S^a$ and $\tilde{\Gamma}_S^b$ means that locally (see figure 2.2)

$$\begin{cases} u^a(\Omega^a) \subset \{(y', y_n) : y_n \geqslant \varphi^c(y'), y' \in U\}, \\ u^b(\Omega^b) \subset \{(y', y_n) : y_n \leqslant \varphi^c(y'), y' \in U\}, \end{cases}$$

where we have set $y = x + u^\iota(x)$, and our contact condition is therefore

$$\varphi^\iota(x') + u_n^\iota(x') = y_n = \varphi^c(y') = \varphi^c(x' + u^{\iota\prime}(x')).$$

2.6. There are many ways to derive this condition. See [6, §1] for the approach this is inspired on, [22, §6.8] for another where the simplification is made that the bodies are supposed joined together and [17, §2.1.2] for a more explicit derivation in the case of $\mathbb{R}^2$.

2.7. See the notation and conventions.

We may read these expressions as follows. Take (the image of) a point x' on Γ_S^ι and displace it with the "vertical displacement" of the solution: $\varphi^\iota(x') + u_n^\iota(x')$. The resulting "height" will be that given by the image φ^c on the contact surface of the point after "lateral" displacement $y' = x' + u^{\iota\prime}(x')$.

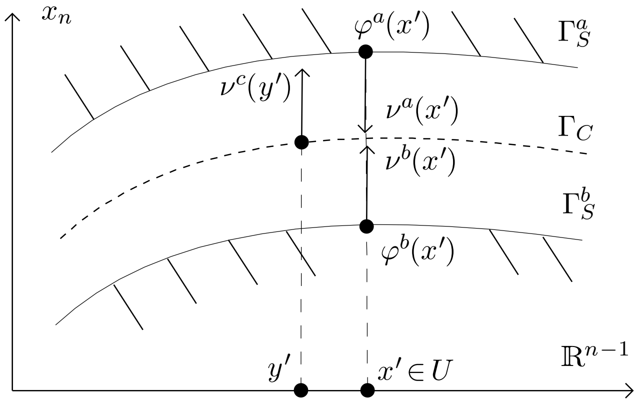


Figure 2.2. Projections onto $U \subset \mathbb{R}^{n-1}$.

The condition that the bodies do not interpenetrate is two-fold:

$$\begin{cases} y_n^a = x_n^a + u_n^a(x') \geqslant \varphi^c(x' + u^{a\prime}(x')), \\ y_n^b = x_n^b + u_n^b(x') \leqslant \varphi^c\big(x' + u^{b\prime}(x')\big). \end{cases}$$

The function φ^c represents how much the contact manifold $\Gamma_C = \{(x', \varphi^c(x')) : x' \in U\}$ differs from the original boundaries Γ_S^ι and small displacements justify ignoring higher order terms of its Taylor expansion $\varphi^c(y') = \varphi^c(x' + u^{\iota\prime}) \simeq \varphi^c(x') + \nabla\varphi^c(x')\, u^{\iota\prime}$, which we use to write all variables in the reference configuration:

$$\begin{cases} x_n^a + u_n^a \geqslant \varphi^c(x') + \nabla\varphi^c(x')\, u^{a\prime}, & \text{for} \quad x^a \in \overline{\Omega}^a, \\ x_n^b + u_n^b \leqslant \varphi^c(x') + \nabla\varphi^c(x')\, u^{b\prime}, & \text{for} \quad x^b \in \overline{\Omega}^b. \end{cases}$$

Notice that we may extend these inequalities to the interior of the bodies thanks to the fact that Γ_C partitions locally the space leaving the deformed body $\tilde{\Omega}^a$ at one side and $\tilde{\Omega}^b$ at the other. We now add both inequalities and use (2.5) to find

$$x_n^a + u_n^a - \nabla\varphi^a(x')\, u^{a\prime} - x_n^b - u_n^b + \nabla\varphi^b(x')\, u^{b\prime} \geqslant 0,$$

which after grouping terms and substituting $x_n^\iota = \varphi^\iota(x')$ leads us to

$$(-\nabla\varphi^a, 1)\, u^a + (\nabla\varphi^b, -1)\, u^b \geqslant \varphi^b - \varphi^a.$$

We observe that the (non unitary) normal vectors to the bodies' boundaries are $\tilde{\nu}^a = (\nabla\varphi^a, -1)$ and $\tilde{\nu}^b = (-\nabla\varphi^b, 1)$ and write finally:

$$u_\nu^a + u_\nu^b \leqslant g, \tag{2.6}$$

where $u_\nu^\iota = u^\iota \cdot \nu^\iota$ denotes the **normal displacement of $\boldsymbol{u^\iota}$ at the boundary** and with $g(x') := [\varphi^a(x') - \varphi^b(x')] \, \|\tilde{\nu}\|^{-1} \geqslant 0$ we model the **normalized initial gap** between the bodies (notice that $\|\tilde{\nu}\| := \|\tilde{\nu}^a\| = \|\tilde{\nu}^b\|$ because of (2.5)). This **kinematical condition** may be further simplified by the assumption that the boundaries are truly parallel to each other ($\nu^a = -\nu^b$), which leads to

$$\nu \, (u^a - u^b) \leqslant g.$$

2.2.2. Boundary stresses.

Condition (2.6) alone is clearly insufficient to determine a solution, and we must accompany it by compatible restrictions on the boundary stresses. We take into account three basic principles:

1. Contact stresses are normal to the surfaces on which they occur and **compressive** (meaning that they "point inwards").
2. In the absence of contact there appears no normal stress.
3. Frictionless contact implies that the tangential component of the stress tensor is zero.

We translate 1. as $\sigma_{ij}^\iota(u^\iota) \, \nu_i^\iota \, \nu_j^\iota \leqslant 0$ on Γ_S^ι. For 2. we have $\sigma_{ij}^\iota(u^\iota) \, \nu_i^\iota \, \nu_j^\iota = 0$ on those points of Γ_S^ι where the strict inequality $\nu^a \, u^a + \nu^b \, u^b < g$ holds. The absence of friction from 3. can be expressed as $\sigma_{ij}^\iota(u^\iota) \, \nu_i^\iota \, \tau_j^\iota = 0$ for any smooth τ^ι which is orthogonal to ν^ι. Finally Newton's third law reads: $\sigma_{ij}^a(u^a) \, \nu_i^a \, \nu_j^a = \sigma_{ij}^b(u^b) \, \nu_i^b \, \nu_j^b$ whenever $\nu^a \, u^a + \nu^b \, u^b = g$

It is important to note that these conditions only make proper sense when we have the common coordinate systems for Γ_S^ι using the open sets $U \subset \mathbb{R}^{n-1}$ introduced in 2.2.1. In particular, the expression "on Γ_S^ι" should be read as "on $\{(x', \varphi^\iota(x'))\colon x' \in U\}$", and this again must be interpreted for all sets U domains of functions whose graphs together are the boundaries of each body. As before, thinking locally we will assume there is just one such set and write for instance for the last condition:

$$\sigma_{ij}^a(u^a(x', \varphi^a(x'))) \, \nu_i^a \, \nu_j^a = \sigma_{ij}^b(u^b(x', \varphi^b(x'))) \, \nu_i^b \, \nu_j^b$$

whenever

$$\nu^a \, u^a(x', \varphi^a(x')) + \nu^b \, u^b(x', \varphi^b(x')) = g(x'), x' \in U.$$

2.3. Problem formulation.

We will now write the set of differential equations together with all the boundary conditions, albeit with strong regularity assumptions which simplify greatly the process of deriving the variational formulation.

We first recall that a bounded open set Ω is called[2.8] **(uniformly) of class $\boldsymbol{C^{\kappa,\lambda}}$** if there exist an open cover $\Omega_0, \ldots, \Omega_r$ of Ω with $d(\Omega_0, \partial\Omega) > 0$ and $\Omega_i \cap \partial\Omega \neq \emptyset$ for $i > 0$, together with an associated uniformly bounded partition of unity, and a uniformly bounded family of $C^{\kappa,\lambda}$ maps $\varphi^i \colon U_i \to \Omega_i$, with U_i open in $\mathbb{R}^{n-1}$, such that possibly after changes of coordinates

$$\Omega_i \cap \Omega \subset \{(x', x_n) \colon x' \in U_i, x_n > \varphi^i(x')\} \text{ and } \Omega_i \cap \partial\Omega = \{(x', \varphi^i(x')) \colon x' \in U_i\}.$$

With all this in mind we can now fully state the boundary problem in its local form:

2.8. This property guarantees among other things the density of infinitely differentiable functions in (the usual) Sobolev spaces and the validity of the (standard) Sobolev embeddings and trace theorems. See [9, Chapter 2].

Problem 2.1. *Let $\iota = a$, b and let $\Omega^\iota \Subset \mathbb{R}^n$ be two open sets with $C^{1,1}$ boundaries Γ^ι, each split in three disjoint, open subsets with $C^{1,1}$ boundaries as well, such that $\overline{\Gamma}^\iota_D \cup \overline{\Gamma}^\iota_N \cup \overline{\Gamma}^\iota_S = \Gamma^\iota$. Assume that the subsets Γ^ι_S lay "almost parallel" but separated in the sense discussed above and that they are contained in the graphs of $C^{1,1}$ maps defined over a common domain $U = \mathring{U} \subset \mathbb{R}^{n-1}$. Let $f^\iota \in \boldsymbol{C}(\overline{\Omega}^\iota)$, $t^\iota \in \boldsymbol{C}(\Gamma^\iota_N)$ and $g \in \boldsymbol{C}(U)$ a nonnegative function. Then find $u = (u^a, u^b) \in \boldsymbol{C}^2(\Omega^a) \cap \boldsymbol{C}(\overline{\Omega}^a) \times \boldsymbol{C}^2(\Omega^b) \cap \boldsymbol{C}(\overline{\Omega}^b)$ such that:*

$$\left\{\begin{array}{rcll} -\operatorname{div} \sigma^\iota(u^\iota) &=& f^\iota & \text{in } \Omega^\iota, \\ \sigma^\iota(u^\iota)\,\nu^\iota &=& t^\iota & \text{on } \Gamma^\iota_N, \\ u^\iota &=& 0 & \text{on } \Gamma^\iota_D, \end{array}\right. \tag{2.7}$$

and

$$\left\{\begin{array}{rclcl} u^a_\nu + u^b_\nu - g &\leqslant& 0 & \text{on} & U, \\ \sigma^\iota_\nu(u^\iota) &\leqslant& 0 & \text{on} & \Gamma^\iota_S, \\ \sigma^a_\nu(u^a) &=& \sigma^b_\nu(u^b) & \text{on} & U, \\ \sigma^\iota_\nu(u^\iota) &=& 0 & \text{whenever} & u^a_\nu + u^b_\nu < g, \\ \sigma^\iota(u^\iota)\,\nu^\iota \cdot \tau^\iota &=& 0 & \text{for all smooth} & \tau^\iota \perp \nu^\iota \text{ on } \Gamma^\iota_S. \end{array}\right. \tag{2.8}$$

where ν^ι is the unit outer normal vector to the boundary Γ^ι and the scalar $\sigma^\iota_\nu := \sigma^\iota_{ij}\,\nu^\iota_j\,\nu^\iota_i$ is the ***normal tension****.*

Other magnitudes defined are the **tangential tension** as the vector $\sigma^\iota_\tau := \sigma^\iota\,\nu^\iota - \sigma^\iota_\nu\,\nu^\iota$ and the **tangential displacement** as $u^\iota_\tau := u^\iota - u^\iota_\nu\,\nu^\iota$.

3. The variational formulation

In this section we weaken the conditions given in problem 2.1 and for this we need the definitions and results from §B. In particular, it is necessary to define the trace space $H_{00}^{1/2}(\Sigma)$, as well as the concept of an ordering in it, together with the weak definitions of normal and tangential components of the trace of a function using generalized formulas for integration by parts. We refer to the aforementioned section and the bibliography cited there. As will be seen, all of these requirements are of technical nature and tend to obscure the actually relevant questions, but the (moderate) level of generality to which we aim makes them inevitable.

Let then $\Omega^\iota \Subset \mathbb{R}^n$, $\iota = a, b$, be open and of class $C^{1,1}$ and $\Gamma_D^\iota, \Gamma_N^\iota, \Gamma_S^\iota, \Sigma^\iota$ be open subsets of Γ^ι of class $C^{1,1}$ as well, such that $\Sigma^\iota = \operatorname{Int}(\Gamma^\iota \backslash \Gamma_D^\iota)$ and $\Gamma_N^\iota, \Gamma_S^\iota \subset \Sigma^\iota$.[3.1, 3.2] Moreover, suppose that the boundaries Γ_S^ι are the graphs of $C^{1,1}$ maps φ^ι with a common domain $U \subset \mathbb{R}^{n-1}$ and take $f^\iota \in \boldsymbol{L}^2(\Omega^\iota), t^\iota \in \boldsymbol{L}^2(\Gamma_N^\iota), g \in H_{00}^{1/2}(U)$. We first define

$$V^\iota := \left\{ v \in \boldsymbol{H}^1(\Omega^\iota) \colon \gamma^\iota(v) = 0 \text{ in } \boldsymbol{H}^{1/2}(\Gamma_D^\iota) \right\},$$

where $\gamma^\iota \colon \boldsymbol{H}^1(\Omega^\iota) \to \boldsymbol{H}^{1/2}(\Gamma^\iota)$ is the trace operator, and as before, we define componentwise: $V := V^a \times V^b$, which is a Hilbert space for the natural scalar product and has norm equivalent to the sum of the norms in each V^ι. Using the normal trace operators $\gamma_{\Sigma_\nu^\iota}^0\colon V \twoheadrightarrow H_{00}^{1/2}(\Sigma^\iota)$ defined in proposition B.6 and the charts φ^a, φ^b we define:[3.3]

$$v_{\nu_{|U}}(x') := \gamma_{\Sigma_\nu^a}^0(v^a)(x', \varphi^a(x')) + \gamma_{\Sigma_\nu^b}^0(v^b)(x', \varphi^b(x')) \in H_{00}^{1/2}(U).$$

With this we introduce the constraints set

$$K := \left\{ v \in V \colon v_{\nu|U} - g \leqslant 0 \text{ on } U \text{ in the sense of } H_{00}^{1/2}(U) \right\}, \tag{3.1}$$

which by lemma 3.6 below is closed and convex.[3.4] Finally let

$$a^\iota(u, v) := \int_{\Omega^\iota} a_{ijkl}^\iota \, \varepsilon_{kl}(u) \, \varepsilon_{ij}(v) \, \mathrm{d}x = \int_{\Omega^\iota} \sigma^\iota(u) \colon \varepsilon(v) \, \mathrm{d}x,$$

$$F^\iota(v) := \int_{\Omega^\iota} f_i^\iota \, v_i \, \mathrm{d}x + \int_{\Gamma_N^\iota} t_i^\iota \, v_i \, \mathrm{d}s_x.$$

In our quest for solutions in some form to problem 2.1 we first derive an alternative weak formulation as the following *variational inequality*:

Problem 3.1. *Find $u = (u^a, u^b) \in K$ such that*

$$a^a(u^a, v^a - u^a) + a^b(u^b, v^b - u^b) \geqslant F^a(v^a - u^a) + F^b(v^b - u^b) \tag{3.2}$$

for every $v = (v^a, v^b) \in K$, or, more succinctly, find $u \in K$ such that:

$$a(u, v - u) \geqslant F(v - u) \text{ for every } v \in K.$$

This formulation is the natural outcome of the usual process of multiplication by test functions and integration by parts and as such it may be seen to be equivalent to the

3.1. The $C^{1,1}$ regularity assumption on the set is in order to have the product of the trace and the normal to the boundary be a function of $H^{1/2}$, something we prove in lemma B.3.

3.2. Here is a crucial point. If we assumed $\Gamma_S^\iota \Subset \Sigma^\iota$ the restricted trace operator $\gamma_{\Gamma_S^\iota}^0$ would be surjective onto $H^{1/2}(\Gamma_S^\iota)$ and the formula for integration by parts of theorem B.8 would hold with $Z = H^{1/2}(\Gamma_S^\iota)$ and we could do without $H_{00}^{1/2}(\Sigma^\iota)$ at a few places.

3.3. Note that here we use that $H_{00}^{1/2}(\Gamma_S^\iota) \subset H_{00}^{1/2}(\Sigma^\iota)$.

3.4. We refer to §B for further details.

boundary problem 2.1 (see theorem 3.1). To motivate problem 3.1 we start with some solution

$$u=(u^a,u^b)\in\tilde{K}$$

of the classical problem, where

$$\tilde{K}:=\left\{v\in\tilde{V}: v_\nu^a+v_\nu^b\leqslant g \text{ on } U\right\},$$

and

$$\tilde{V}:=\left\{v\in\boldsymbol{C}^2(\Omega^a)\cap\boldsymbol{C}(\overline{\Omega}^a)\times\boldsymbol{C}^2(\Omega^b)\cap\boldsymbol{C}(\overline{\Omega}^b): v^\iota_{|\Gamma_D^\iota}=0\right\},$$

then choose some $v=(v^a,v^b)\in\tilde{K}$ and multiply each of the first set of equations in (2.7) by $v^\iota-u^\iota$ and integrate by parts. From

$$-\int_{\Omega^\iota}\operatorname{div}(\sigma^\iota)\cdot(v^\iota-u^\iota)\,\mathrm{d}x=\int_{\Omega^\iota}\sigma^\iota:\varepsilon(v^\iota-u^\iota)\,\mathrm{d}x-\int_{\Gamma^\iota}\sigma^\iota\,\nu^\iota\cdot(v^\iota-u^\iota)\,\mathrm{d}s_x$$

follows:

$$\begin{aligned}\int_{\Omega^\iota}\sigma^\iota:\varepsilon(v^\iota-u^\iota)\,\mathrm{d}x &= \int_{\Omega^\iota}f^\iota\cdot(v^\iota-u^\iota)\,\mathrm{d}x+\int_{\Gamma_N^\iota}\sigma^\iota\,\nu^\iota\cdot(v^\iota-u^\iota)\,\mathrm{d}s_x\\ &\quad+\int_{\Gamma_D^\iota}\sigma^\iota\,\nu^\iota\cdot(v^\iota-u^\iota)\,\mathrm{d}s_x+\int_{\Gamma_S^\iota}\sigma^\iota\,\nu^\iota\cdot(v^\iota-u^\iota)\,\mathrm{d}s_x.\end{aligned}$$

Of the three boundary integrals the second vanishes because of the Dirichlet condition and the first one equals $\int_{\Gamma_N^\iota}t^\iota\cdot(v^\iota-u^\iota)\,\mathrm{d}s_x$. In order to study the third we decompose the vector $\sigma^\iota\,\nu^\iota$ in the normal and tangential tensions: $\sigma^\iota\,\nu^\iota=\sigma_\tau^\iota+\sigma_\nu^\iota\,\nu^\iota$ with $\sigma_\nu^\iota=\sigma_{ij}^\iota\,\nu_j^\iota\,\nu_i^\iota$, and see that

$$\int_{\Gamma_S^\iota}\sigma^\iota\,\nu^\iota\cdot(v^\iota-u^\iota)\,\mathrm{d}s_x=\int_{\Gamma_S^\iota}\sigma_\tau^\iota\cdot(v^\iota-u^\iota)+\sigma_\nu^\iota\nu^\iota\cdot(v^\iota-u^\iota)\,\mathrm{d}s_x=\int_{\Gamma_S^\iota}\sigma_\nu^\iota\,v_\nu^\iota-\sigma_\nu^\iota\,u_\nu^\iota\,\mathrm{d}s_x$$

because the term with σ_τ^ι is zero due to the lack of friction. Using the common representation with U of the boundaries Γ_S^ι we add these integrals for both bodies: we let $\psi^\iota(x'):=(x',\varphi^\iota(x'))$, $x'\in U$ be the charts for Γ_S^ι and define

$$\omega^\iota(x'):=|\det(\partial_1\,\psi^\iota(x'),\ldots,\partial_{n-1}\,\psi^\iota(x'),\nu^\iota(x'))|,$$

then the surface integrals are

$$\int_{\Gamma_S^\iota}(\sigma_\nu^\iota\,v_\nu^\iota-\sigma_\nu^\iota\,u_\nu^\iota)(x)\,\mathrm{d}s_x=\int_U(\sigma_\nu^\iota\,v_\nu^\iota-\sigma_\nu^\iota\,u^\iota)(\psi^\iota(x'))\,\omega^\iota(x')\,\mathrm{d}x'.$$

However, relation (2.5) implies $\omega(x'):=\omega^a(x')=\omega^b(x')$ for every $x'\in U$. Using also the facts from (2.8) that $\sigma_\nu^\iota=0$ if $u_\nu^a+u_\nu^b<g$ and $\sigma_\nu^a\circ\psi^a(x')=\sigma_\nu^b\circ\psi^b(x')=:\sigma_\nu(x')$ on U we have

$$\begin{aligned}&\int_U\left[(\sigma_\nu^a\,v_\nu^a-\sigma_\nu^a\,u_\nu^a)\circ\psi^a+\left(\sigma_\nu^b\,v_\nu^b-\sigma_\nu^b\,u_\nu^b\right)\circ\psi^b\right]\omega\,\mathrm{d}x'\\ &\quad=\int_{\{u_\nu^a+u_\nu^b=g\}}\left[(\sigma_\nu^a\,v_\nu^a-\sigma_\nu^a\,u_\nu^a)\circ\psi^a+\left(\sigma_\nu^b\,v_\nu^b-\sigma_\nu^b\,u_\nu^b\right)\circ\psi^b\right]\omega\,\mathrm{d}x'\\ &\quad=\int_{\{u_\nu^a+u_\nu^b=g\}}\left[\sigma_\nu\left(v_\nu^a\circ\psi^a+v_\nu^b\circ\psi^b\right)-\sigma_\nu\left(u_\nu^a\circ\psi^a+u_\nu^b\circ\psi^b\right)\right]\omega\,\mathrm{d}x'\\ &\quad=\int_{\{u_\nu^a+u_\nu^b=g\}}\left[\sigma_\nu\left(v_\nu^a+v_\nu^b-g\right)-\sigma_\nu\left(u_\nu^a+u_\nu^b-g\right)\right]\omega\,\mathrm{d}x'\\ &\quad=\int_{\{u_\nu^a+u_\nu^b=g\}}\left[\sigma_\nu\left(v_\nu^a+v_\nu^b-g\right)\right]\omega\,\mathrm{d}x'\\ &\quad\geqslant 0,\end{aligned}$$

the last step due to $\sigma_\nu^\iota \leqslant 0$ on Γ_S^ι and $v \in \tilde{K}$. Adding the equations for both bodies we arrive at

$$\begin{aligned}\sum_\iota \int_{\Omega^\iota} \sigma^\iota : \varepsilon(v^\iota - u^\iota)\,\mathrm{d}x &= \sum_\iota \int_{\Omega^\iota} f^\iota \cdot (v^\iota - u^\iota)\,\mathrm{d}x + \sum_\iota \int_{\Gamma_N^\iota} t^\iota \cdot (v^\iota - u^\iota)\,\mathrm{d}s_x \\ &\quad + \sum_\iota \int_{\Gamma_S^\iota} \sigma^\iota \nu^\iota \cdot (v^\iota - u^\iota)\,\mathrm{d}s_x \\ &\geqslant \sum_\iota \int_{\Omega^\iota} f^\iota \cdot (v^\iota - u^\iota)\,\mathrm{d}x + \sum_\iota \int_{\Gamma_N^\iota} t^\iota \cdot (v^\iota - u^\iota)\,\mathrm{d}s_x,\end{aligned}$$

or, in the notation of problem 3.1:

$$a(u, v-u) \geqslant F(v-u) \text{ for every } v \in \tilde{K}.$$

It is clear that this inequality still makes sense for more general functions, e.g. for those in K, as long as the trace operators, orderings and other details are properly dealt with.

In the next theorem we prove that, once this is done, we are left with a problem which to some extent contains the original one. We note that in addition to the technical requirements already mentioned, the conditions of problem 2.1 must hold in open subsets of U in order to be able to use density arguments with test functions.

Theorem 3.1. *Any solution to problem 3.1 is a solution to the following weak form of problem 2.1:*

> ***Problem 3.2.*** *Let $\iota = a,\, b$ and let $\Omega^\iota \Subset \mathbb{R}^n$ be two open sets with $C^{1,1}$ boundaries Γ^ι, each split in three disjoint, open subsets with $C^{1,1}$ boundaries as well, such that $\overline{\Gamma}_D^\iota \cup \overline{\Gamma}_N^\iota \cup \overline{\Gamma}_S^\iota = \Gamma^\iota$. Let $\Sigma^\iota = \operatorname{Int}\Gamma^\iota \backslash \Gamma_D$. Assume that the subsets Γ_S^ι lay "almost parallel" but separated in the sense discussed in §2.2.1 and that there are charts with common domain $U = \mathring{U} \subset \mathbb{R}^{n-1}$ covering sets containing the Γ_S^ι. Let $f^\iota \in \boldsymbol{L}^2(\overline{\Omega}^\iota), t^\iota \in \boldsymbol{L}^2(\Gamma_S^\iota)$ and $g \in H_{00}^{1/2}(U)$ a nonnegative function. Then find $u = (u^a, u^b) \in \boldsymbol{H}^1(\Omega^a) \times \boldsymbol{H}^1(\Omega^b)$ such that:*
>
> $$\begin{cases} -\operatorname{div}\sigma^\iota(u^\iota) &= f^\iota \quad \text{in } \boldsymbol{L}^2(\Omega^\iota), \\ \sigma^\iota(u^\iota)\,\nu^\iota &= t^\iota \quad \text{in } \boldsymbol{L}^2(\Gamma_N^\iota), \\ u^\iota &= 0 \quad \text{in } \boldsymbol{H}^{1/2}(\Gamma_D^\iota), \end{cases}$$
>
> *and*
>
> $$\begin{cases} u_\nu^a + u_\nu^b - g \leqslant 0 & \text{on} \quad U \text{ in } H_{00}^{1/2}(U), \\ \sigma_\nu^\iota(u^\iota) \leqslant 0 & \text{on} \quad \Gamma_S^\iota \text{ in } H_{00}^{-1/2}(\Sigma^\iota), \\ \sigma_\nu^a(u^a) = \sigma_\nu^b(u^b) & \text{on} \quad \Gamma_S^\iota \text{ in } H_{00}^{-1/2}(\Sigma^\iota), \\ \sigma_\nu^\iota(u^\iota) = 0 & \text{on} \quad U' \text{ in } H_{00}^{-1/2}(\Sigma^\iota) \\ & \text{whenever} \quad u_\nu^a + u_\nu^b < g \text{ in } U' \Subset U \text{ open}, \\ \sigma_\tau^\iota(u^\iota) = 0 & \text{on} \quad \Gamma_S^\iota \text{ in } H_{00}^{-1/2}(\Sigma^\iota). \end{cases}$$

Proof. The fundamental tool in this proof will be the formula of integration by parts from §B.4. Assume $u = (u^a,\, u^b) \in K$ solves problem 3.1 and choose $\psi = (\psi^a,\, \psi^b)$ with $\psi^\iota \in \mathcal{D}^\iota := C_0^\infty(\Omega^\iota)$. Then $v := u \pm \psi \in K$ and equation (3.2) reads

$$\pm \int_{\Omega^a} \sigma^a(u^a) : \varepsilon(\psi^a)\,\mathrm{d}x \pm \int_{\Omega^b} \sigma^b(u^b) : \varepsilon(\psi^b)\,\mathrm{d}x \geqslant \pm \int_{\Omega^a} f^a \cdot \psi^a\,\mathrm{d}x \pm \int_{\Omega^b} f^b \cdot \psi^b\,\mathrm{d}x.$$

This means that we have an equality for every $\psi \in \mathcal{D}^a \times \mathcal{D}^b$ and setting first $\psi^a = 0$ then $\psi^b = 0$ we have in fact equalities in the sense of distributions on $\mathcal{D}^\iota$. We differentiate these distributions to get

$$\langle [\sigma^\iota], \varepsilon(\psi^\iota)\rangle_{\mathcal{D}^{\iota\prime}\times\mathcal{D}^\iota} = \langle [f^\iota], \psi^\iota\rangle_{\mathcal{D}^{\iota\prime}\times\mathcal{D}^\iota} \Rightarrow \langle -\operatorname{div}[\sigma^\iota], \psi^\iota\rangle_{\mathcal{D}^{\iota\prime}\times\mathcal{D}^\iota} = \langle [f^\iota], \psi^\iota\rangle_{\mathcal{D}^{\iota\prime}\times\mathcal{D}^\iota},$$

but $f^\iota \in \boldsymbol{L}^2(\Omega^\iota)$ so in fact

$$-\operatorname{div}\sigma^\iota(u^\iota) = f^\iota \text{ in } \boldsymbol{L}^2(\Omega^\iota).$$

Knowing this, we can apply corollary B.9 to integrate by parts. We choose ψ^ι as the extension of some function in $C_0^\infty(\Gamma_N^\iota)$ by zero to Γ^ι, then differentiably to all of $\overline{\Omega}$, and again the functions $v := u \pm \psi \in K$ and we can compute:

$$\begin{aligned}\int_{\Omega^\iota} \sigma^\iota : \varepsilon(\psi^\iota)\,\mathrm{d}x &= \langle \sigma^\iota \nu^\iota, \gamma(\psi^\iota)\rangle_{\boldsymbol{H}^{-1/2}(\Gamma^\iota)} - \int_{\Omega^\iota} \operatorname{div}\sigma^\iota \cdot \psi^\iota\,\mathrm{d}x \\ &= \int_{\Omega^\iota} f^\iota \cdot \psi^\iota\,\mathrm{d}x + \int_{\Gamma_N^\iota} t^\iota\,\psi^\iota\,\mathrm{d}s_x.\end{aligned}$$

Cancelling the terms due to $-\operatorname{div}\sigma^\iota(u^\iota) = f^\iota$ in $\boldsymbol{L}^2$ we have

$$\langle \sigma^\iota \nu^\iota, \gamma(\psi^\iota)\rangle_{\boldsymbol{H}^{-1/2}(\Gamma^\iota)} = \int_{\Gamma_N^\iota} t^\iota\,\psi^\iota\,\mathrm{d}s_x,$$

and because of the choice of ψ^ι arbitrarily in a dense subset of $\boldsymbol{L}^2(\Gamma_N)$ this implies that the linear form $\sigma^\iota\,\nu^\iota \in \boldsymbol{H}^{-1/2}(\Gamma^\iota)$ acts on $\boldsymbol{L}^2(\Gamma_N^\iota)$ via the representation given by t^ι and consequently

$$\sigma^\iota\,\nu^\iota = t^\iota \text{ in } \boldsymbol{L}^2(\Gamma_N^\iota).$$

The Dirichlet condition is included in the set V so there is nothing to check. The same happens with $u_\nu^a + u_\nu^b - g \leqslant 0$ on U in $H_{00}^{1/2}(U)$.

Let now ψ^ι be a differentiable extension to $\mathbb{R}^n$ of some function in $C_0^\infty(\Gamma_S^\iota)$ such that $\psi^\iota \cdot \nu^\iota \leqslant 0$, $\psi^\iota_{|\Gamma_D^\iota} = 0$, $\psi^\iota_{|\Gamma_N^\iota} = 0$ and $\gamma^0_{\Sigma_\tau^\iota}(\psi^\iota) = 0$. Define $v := u + \psi$, which is in K, and integrate by parts using corollary B.10. After cancelling the terms we already discussed we find

$$\langle \sigma_\nu^\iota, \gamma^0_{\Sigma_\nu^\iota}(\psi^\iota)\rangle_{H_{00}^{-1/2}(\Sigma^\iota)} + \langle \sigma_\tau^\iota, \gamma^0_{\Sigma_\tau^\iota}(\psi^\iota)\rangle_{\boldsymbol{H}_{00,\tau}^{-1/2}(\Sigma^\iota)} \geqslant \int_{\Gamma_N^\iota} t^\iota\,\psi^\iota\,\mathrm{d}s_x. \tag{3.3}$$

The second and third terms are zero because of our choice of ψ^ι. Furthermore, its arbitrariness and the surjectivity of $\gamma^0_{\Sigma_\nu^\iota}$ imply, together with $\psi^\iota \cdot \nu^\iota \leqslant 0$, that (see corollary B.7)

$$\sigma_\nu^\iota \leqslant 0 \text{ on } \Gamma_S^\iota \text{ in } H_{00}^{-1/2}(\Sigma).$$

Choosing now $0 \not\equiv \psi^\iota \perp \nu^\iota$ (that is $\gamma^0_{\Sigma_\nu^\iota}(\psi^\iota) = 0$ and $\gamma^0_{\Sigma_\tau^\iota}(\psi^\iota) \neq 0$) the functions $v := u \pm \psi$ are again in K and we have an equality in (3.3), now with the first and third terms vanishing. Consequently we have the last item in the list of boundary conditions:

$$\sigma_\tau^\iota = 0 \text{ on } \Gamma_S^\iota \text{ in } H_{00}^{-1/2}(\Sigma).$$

Now take $\lambda^\iota \in C_0^\infty(\Gamma_S^\iota)$ with $\lambda^a = -\lambda^b$ and extending it by zero define $\psi^\iota := \lambda^\iota \cdot \nu^\iota$ on all of Γ^ι. Then $\psi^\iota_{|\Gamma_D^\iota} = \psi^\iota_{|\Gamma_N^\iota} = 0$, $\gamma^0_{\Sigma_\tau^\iota}(\psi^\iota) = 0$ and $v := u \pm \psi \in K$ because

$$v_\nu^a + v_\nu^b = u_\nu^a + u_\nu^b \pm \lambda^a\,\nu^a \cdot \nu^a \pm \lambda^b\,\nu^b \cdot \nu^b = u_\nu^a + u_\nu^b \leqslant g.$$

As before we obtain an equality from (3.3) where the second and third terms vanish but now we cannot set one of the ψ^ι to zero to isolate the remaining sum in ι because we chose $\lambda^a = -\lambda^b$. Therefore we have:

$$\begin{aligned} 0 &= \langle \sigma_\nu^a, \psi_\nu^a \rangle_{H_{00}^{-1/2}(\Sigma^a)} + \left\langle \sigma_\nu^b, \psi_\nu^b \right\rangle_{H_{00}^{-1/2}(\Sigma^b)} \\ &= \langle \sigma_\nu^a, \lambda^a \nu^a \cdot \nu^a \rangle_{H_{00}^{-1/2}(\Sigma^a)} - \left\langle \sigma_\nu^b, \lambda^a \nu^b \cdot \nu^b \right\rangle_{H_{00}^{-1/2}(\Sigma^b)}. \end{aligned}$$

Because $\nu^\iota \cdot \nu^\iota = 1$ and we chose λ^ι arbitrarily but having compact support in U, this implies

$$\sigma_\nu^a = \sigma_\nu^b \text{ in } H_{00}^{-1/2}(U).$$

Suppose finally that $u_\nu^a + u_\nu^b < g$ on $U' \Subset U$ and take scalar functions $\eta^\iota \in C_0^\infty(U)$ such that $\eta^\iota \neq 0$ in U'. Defining $\psi^\iota := \eta^\iota \nu^\iota$, extend them differentiably by zero in Ω^ι so that we have $\psi^\iota_{|\Gamma_D^\iota} = \psi^\iota_{|\Gamma_N^\iota} = 0$ and $\gamma^0_{\Sigma_\tau^\iota}(\psi^\iota) = 0$. Let $\delta = \inf \left\{ g - \left(u_\nu^a + u_\nu^b \right) \right\}$ and $\mu = \delta / (2 \max_\iota \|\eta^\iota\|_\infty)$. Then

$$\sum_\iota u_\nu^\iota \pm \mu \, \psi^\iota \cdot \nu^\iota = \sum_\iota u_\nu^\iota \pm \mu \, \eta^\iota < g$$

and consequently $v := u \pm \mu \, \psi \in K$. Again we have an equality in (3.3) with two of the terms vanishing:

$$\langle \sigma_\nu^\iota, \mu \, \gamma^0_{\Sigma_\nu^\iota}(\psi^\iota) \rangle_{H_{00}^{-1/2}(\Sigma^\iota)} = 0 \Longrightarrow \langle \sigma_\nu^\iota, \eta^\iota \rangle_{H_{00}^{-1/2}(\Sigma^\iota)} = 0.$$

Because the η^ι were arbitrary it follows

$$\sigma_\nu^\iota = 0 \text{ on } \Upsilon' \text{ in } H_{00}^{-1/2}(\Sigma^\iota),$$

where Υ' is the portion of Σ^ι corresponding to U'. □

We finally introduce another formulation in terms of *minimization of the total energy* which, as we show below in theorem 3.2,, has problem 3.1 as its "Euler-Lagrange inequality":

Problem 3.3. *Let $I: K \to \mathbb{R}$ be the* ***total energy*** *defined for $v = (v^a, v^b) \in K$ defined as*

$$I(v) := I^a(v^a) + I^b(v^b), \tag{3.4}$$

where

$$I^\iota(v^\iota) := \frac{1}{2} a^\iota(v^\iota, v^\iota) - F^\iota(v^\iota).$$

Then find $u = (u^a, u^b) \in K$ such that

$$I(u) = \min_{v \in K} I(v).$$

The next theorem shows that minima of I characterize solutions to the variational inequality. However, we will not be able to exploit this fact other than when we are able to prove existence of a minimum, and for this we will need coercivity of I. We discuss these matters in §3.2 and §A below.

Theorem 3.2. *A function $u \in K$ is a solution to problem 3.1 if and only if it is a minimum for the energy functional (3.4) over K.*

Proof. By proposition 3.8 below the functionals I^ι are Gâteaux differentiable and we have

$$\langle DI(u), v-u\rangle = a(u, v-u) - F(v-u).$$

Now, if u is a minimum then by proposition A.4: $\langle DI(u), v-u\rangle \geqslant 0$ for all $v \in K$ and we just apply the preceding equation. Reciprocally if u solves problem 3.1 we use the convexity of I and proposition A.2:

$$I(v) - I(u) \geqslant \langle DI(u), v-u\rangle = a(u, v-u) - F(v-u) \geqslant 0 \text{ for all } v \in K$$

and this concludes the proof. □

3.1. Rigid displacements.

A rigid displacement, i.e. one without deformation, is the composition of a rotation and a translation: $r(x) = b + Q\,x$, where Q is an orthogonal matrix. Because we are considering infinitesimal motions everywhere, we may exclude the symmetries from the admissible displacements to ensure that Q does not have the eigenvalue -1. Consequently $I + Q$ is invertible and we may define $R := (I - Q)\,(I + Q)^{-1}$. From this matrix we can recover Q using the inverse transform $R \mapsto (I + R)\,(I - R)^{-1}$, which bears the name of **Cayley transform**.

The point in this is that R is antisymmetric, hence determined by n coefficients (the dimension of the space) and every rigid displacement of the kind considered may be described with only two vectors. In $\mathbb{R}^3$ this means

$$r(x) = b + r \times x = b + Rx,$$

where $r \times x$ is of course the usual vector product and the matrix is

$$R = \begin{pmatrix} 0 & -r_3 & r_2 \\ r_3 & 0 & -r_1 \\ -r_2 & r_1 & 0 \end{pmatrix}.$$

The set of **rigid displacements** described in this way will be

$$\mathcal{R} := \{r : \Omega^\iota \to \mathbb{R}^n : r(x) = b + Rx\}.$$

In the context of linear elasticity, the antisymmetry of the matrix means that the bilinear form associated to the elastic potential vanishes exactly on $\mathcal{R}$. In order to prove this, we first need a lemma:

Lemma 3.3. *Let Ω be open and $C^{0,1}$ and let $f \in \boldsymbol{L}^1_{\text{loc}}(\Omega)$ be such that $[f]_{,i} = 0$ as a distribution. Then f is a function independent of x_i.*

Proof. Let $\Omega_\square \Subset \Omega$ be an open square in Ω such that $x_i \in [l_1, l_2]$ for all $x \in \Omega_\square$ and let $\delta > 0$ be small enough that $l_1 + \delta < l_2 - \delta$. Let $h : \mathbb{R} \to \mathbb{R}$ be C^∞ and such that $h(x) = 0$ for $x < l_1 + \delta$ and $h(x) = 1$ for $x > l_2 - \delta$. Note that $h' \in C_0^\infty(\mathbb{R})$ and define for arbitrary $\psi \in \mathcal{D}(\Omega_\square)$:

$$\phi(x_1, \ldots, x_n) := \int_{-\infty}^{x_i} \psi(x_1, \ldots, y, \ldots, x_n)\,\mathrm{d}y - h(x_i) \int_{\mathbb{R}} \psi(x_1, \ldots, y, \ldots, x_n)\,\mathrm{d}y.$$

This function is infinitely differentiable and has compact support in $\Omega_\square$ by construction: if $x_i < l_1 + \delta$ then $\phi(x) = \int_{-\infty}^{x_i} \psi(x_1, \ldots, y, \ldots, x_n)\,\mathrm{d}y$ has compact support because ψ does and if $x_i > l_2 - \delta$ and is outside the support of ψ, then $\phi(x) = 0$.

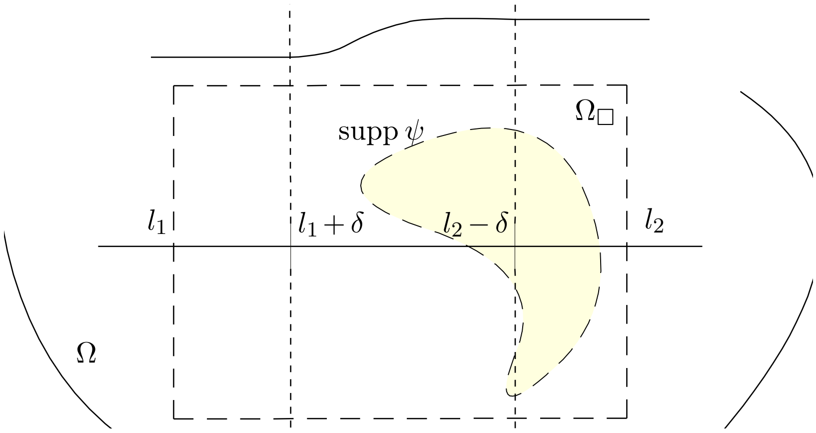


Figure 3.1. The sets $\Omega_\square$.

Let $\Psi(x_1,\ldots,\hat{x}_i,\ldots,x_n)=\int_{\mathbb{R}}\psi(x_1,\ldots,y,\ldots,x_n)\,\mathrm{d}y$, where the circumflex means that Ψ is independent of x_i. Because $\mathcal{D}(\Omega_\square)\subset\mathcal{D}(\Omega)$ we have by hypothesis (note that $h'\,\Psi\in\mathcal{D}(\Omega_\square)$)

$$0=\langle[f]_{,i},\phi\rangle_{\mathcal{D}'(\Omega)}=-\langle[f],\phi_{,i}\rangle_{\mathcal{D}'(\Omega)}=-\langle[f],\psi\rangle_{\mathcal{D}'(\Omega)}+\langle[f],h'\,\Psi\rangle_{\mathcal{D}'(\Omega)}.$$

We may write this as

$$\begin{aligned}
\int_{\Omega_\square} f\psi\,\mathrm{d}x &= \int_{\Omega_\square} f(x_1,\ldots,x_n)\,h'(x_i)\,\Psi(x_1,\ldots,\hat{x}_i,\ldots,x_n)\,\mathrm{d}x_1\ldots\mathrm{d}x_n\\
&= \int_{\mathbb{R}^{n-1}}\Psi(x_1,\ldots,\hat{x}_i,\ldots,x_n)\underbrace{\int_{\mathbb{R}} f(x_1,\ldots,x_n)\,h'(x_i)\,\mathrm{d}x_i}_{=:g(x_1,\ldots,\hat{x}_i,\ldots,x_n)}\mathrm{d}x_1\ldots\mathrm{d}\hat{x}_i\ldots\mathrm{d}x_n\\
&= \int_{\mathbb{R}^{n-1}}\int_{\mathbb{R}}\psi\,\mathrm{d}x_i\,g(x_1,\ldots,\hat{x}_i,\ldots,x_n)\,\mathrm{d}x_1\ldots\mathrm{d}\hat{x}_i\ldots\mathrm{d}x_n\\
&= \int_{\mathbb{R}^n} g(x_1,\ldots,\hat{x}_i,\ldots,x_n)\,\psi\,\mathrm{d}x_1\ldots\mathrm{d}x_n,
\end{aligned}$$

where we repeatedly used Fubini-Tonelli's theorem thanks to $f\in L^1_{\mathrm{loc}}(\Omega)$ and the compact support of h' and ψ. Because $\psi\in\mathcal{D}(\Omega_\square)$ was arbitrary the last equality means that f is independent of x_i on $\Omega_\square$. But Ω is open and $C^{0,1}$ and thus any $x\in\Omega$ has such a square around it which is contained in Ω, meaning that f is independent of x_i a.e. in Ω. □

Proposition 3.4. *Let $\Omega\Subset\mathbb{R}^n, n\leqslant 3$ and $w\in\boldsymbol{H}^1(\Omega)$. Then $\varepsilon_{ij}(w)=0$ if and only if w is a rigid displacement.*

Proof. If w is a rigid displacement, its differential R is an antisymmetric matrix, whose symmetric part is then zero, i.e. $\varepsilon_{ij}(w)=0$ for all i,j. To prove the converse we let $n=3$, the other cases being simpler versions of this one, and suppose that $\varepsilon_{ij}(w)=\frac{1}{2}(w_{i,j}+w_{j,i})=0$. Then, in $L^2(\Omega)$:

$$w_{i,j}=-w_{j,i}\text{ for all }i,j\text{ and }w_{i,i}=0\text{ for all }i, \tag{3.5}$$

the latter setting $i=j$. We differentiate again, in the sense of distributions: $\varepsilon_{ij,k}(w)=0$ in $\mathcal{D}'(\Omega)$ and we use the equality of mixed partial derivatives for distributions to have

$$w_{i,jk}=-w_{j,ik}=-w_{j,ki}=w_{k,ji}\text{ for all }i,j,k\text{ in }\mathcal{D}'(\Omega).$$

Choosing $j = k$ this implies $w_{i,jj} = -w_{j,ij} = -w_{j,ji} = 0$ in $\mathcal{D}'(\Omega)$ by (3.5), that is: $w_{i,j}$ is a function independent of x_j by lemma 3.3. But it is also independent of the variable x_i because $w_{i,ji} = w_{i,ij} = 0$, so we have:

$$w_{i,j}(x_1, \ldots, x_n) = w_{i,j}(x_1, \ldots, \hat{x}_i, \ldots, \hat{x}_j, \ldots, x_n),$$

where the circumflex means the omission of a particular variable. Differentiating again and recalling that $n = 3$ we see that the second derivatives are constant distributions: $w_{i,jk} = c_{ijk}$ in $\mathcal{D}'(\Omega)$ and because of the relationships $w_{i,j} = -w_{j,i}$ we conclude that the constants c_{ijk} are all zero: $w_{i,jk} = -w_{j,ik} = -w_{j,ki} = w_{k,ji} = w_{k,ij} = -w_{i,kj} = -w_{i,jk}$. Consequently

$$w_{i,j} = c_{ij} = -c_{ji} = -w_{j,i} \text{ in } L^2(\Omega),$$

which means that the differential matrix $Dw \in L^2(\Omega)^{n \times n}$ is constant and antisymmetric, that is: w is a rigid displacement. $\square$

The following corollary is an obvious consequence of proposition 3.4.

Corollary 3.5. *Let $r \in \mathcal{R} \cap \boldsymbol{H}^1(\Omega^\iota)$, and $\mathcal{E}$ be defined as in §C. Then $\mathcal{E}(r) = 0$ and consequently $a^\iota(r, \cdot) = a^\iota(\cdot, r) = 0$.*

3.2. Existence and uniqueness.

As explained in §A, the key properties to have in order to prove existence and uniqueness of solution when working with problem 3.3 are convexity, Gâteaux differentiability and coercivity of the energy functional. The latter is by far the most complicated one and using Korn's inequality we must distinguish between the case with displacements set on some subset of the boundary ($\operatorname{meas} \Gamma_D^\iota > 0$) and the pure tractions/contact case ($\operatorname{meas} \Gamma_D^\iota = 0$). In the latter situation, the energy functional fails to be coercive and we are left with uniqueness up to a displacement in a subset of $\mathcal{R}$.

Lemma 3.6. *The set K of (3.1) is closed and convex.*

Proof. The normal trace map is continuous and the proper cone $\mathcal{C}_+$ is closed (seen in proposition B.2), hence K is closed. Its convexity is a consequence of the linearity of weak convergence, by exactly the same argument as in the proof of that proposition. $\square$

Proposition 3.7. *The functional I^ι is convex in K. If $\operatorname{meas} \Gamma_D > 0$, then it is strictly convex.*

Proof. Let $\theta \in (0, 1)$ and $u \neq v$ be in K. Then by linearity and the positive semi-definiteness of the quadratic form $a^\iota(\cdot, \cdot)$:

$$\begin{aligned}
&\frac{1}{2} a^\iota(\theta\, u + (1-\theta)\, v, \theta\, u + (1-\theta)\, v) \\
&\qquad = \frac{1}{2} a^\iota(\theta(u - v) + v, \theta(u - v) + v) \\
&\qquad = \frac{\theta^2}{2} a^\iota(u - v, u - v) + a^\iota(v, \theta\,(u - v)) + \frac{1}{2} a^\iota(v, v) \\
&\qquad \leqslant \frac{\theta}{2} a^\iota(u, u) - \theta\, a^\iota(u, v) + \frac{\theta}{2} a^\iota(v, v) + a^\iota(v, \theta\,(u - v)) + \frac{1}{2} a^\iota(v, v) \\
&\qquad = \frac{\theta}{2} a^\iota(u, u) + \frac{1-\theta}{2} a^\iota(v, v).
\end{aligned}$$

And therefore:

$$\begin{aligned} I^\iota(\theta\, u + (1-\theta)\, v) &\leqslant \frac{\theta}{2}\, a^\iota(u,u) + \frac{1-\theta}{2}\, a^\iota(v,v) - \theta\, F^\iota(u) - (1-\theta)\, F^\iota(v) \\ &= \theta\, I^\iota(u) + (1-\theta)\, I^\iota(v). \end{aligned}$$

If $\operatorname{meas}\Gamma_D > 0$, we know by lemma 3.9 below that a is positive definite and in the third step above, we may write a strict inequality, thus having strict convexity. □

Proposition 3.8. *The functional I^ι is Gâteaux differentiable in $\boldsymbol{H}^1(\Omega^\iota)$ with derivative:*

$$\langle D I^\iota(u), v\rangle = a^\iota(u,v) - F^\iota(v) \text{ for every } v \in \boldsymbol{H}^1(\Omega^\iota).$$

Proof. We must check whether the directional derivative can be represented with a linear functional continuous on $\boldsymbol{H}^1(\Omega)$ (see (A.1)). To this end let $\lambda > 0$, u, $v \in \boldsymbol{H}^1(\Omega^\iota)$ and compute:

$$\begin{aligned} &\frac{1}{\lambda}\,[I^\iota(u+\lambda\, v) - I^\iota(u)] \\ &\quad= \frac{1}{\lambda}\left[\frac{1}{2}\, a^\iota(u+\lambda\, v, u+\lambda\, v) - \frac{1}{2}\, a^\iota(u,u) - F^\iota(u+\lambda\, v) + F^\iota(u)\right] \\ &\quad= \frac{1}{\lambda}\left[a^\iota(u,\lambda\, v) + \frac{\lambda^2}{2}\, a^\iota(v,v) - F^\iota(\lambda\, v)\right] \\ &\quad= a^\iota(u,v) + \frac{\lambda}{2}\, a^\iota(v,v) - F^\iota(v) \\ &\quad\underset{\lambda\to 0}{\longrightarrow} a^\iota(u,v) - F^\iota(v) \\ &\quad=: D I^\iota(u)(v). \end{aligned}$$

The functions $D I^\iota(u)$ so defined are clearly linear in v and the continuity of $a^\iota(\cdot,\cdot)$ and $F^\iota(\cdot)$ imply that they are also continuous on $\boldsymbol{H}^1(\Omega^\iota)$. □

The functional $\boldsymbol{I^\iota}$ is coercive. As announced, Korn's inequality (cf. §C) is the central piece in the proof of coercivity of the energy functional (3.4) but its applicability depends heavily on the kernel of the quadratic forms a^ι, which is the set $\mathcal{R}$ of rigid displacements as seen in corollary 3.5. This kernel relates to the physical fact that the elastic energy of the bodies is not altered by rigid displacements like rotations or shifts, so we must exclude them somehow from the admissible ones if we are to expect coercivity, or resort to other methods. This can be achieved through Dirichlet boundary conditions because fixing the displacement on any subset of the boundary with positive measure effectively excludes any non-zero rigid movement from taking place. Therefore we must distinguish between the case where this is done, i.e. $\operatorname{meas}\Gamma_D^\iota > 0$, and the one where the bodies are not fixed at all, which is $\operatorname{meas}\Gamma_D^\iota = 0$.

The case $\boldsymbol{\operatorname{meas}\Gamma_D^\iota > 0}$. This case is handled in the next lemma thanks to corollary C.2 to Korn's inequality, and the assumption that $\operatorname{meas}\Gamma_D^\iota > 0$ is essential to conclude that the only admissible rigid displacement is the zero function. This allows us to use right away the tools of convex minimization to prove existence and uniqueness.

Lemma 3.9. *Assume that Γ_D^ι has positive measure and that the ellipticity condition (2.2) holds for the coefficients a_{ijkl}^ι. Then the bilinear forms $a^\iota(u,v) = \int_\Omega a_{ijkl}^\iota(x)\, \varepsilon_{kl}(u)\, \varepsilon_{ij}(v)\, \mathrm{d}x$ are elliptic on*

$$V^\iota := \{v \in \boldsymbol{H}^1(\Omega^\iota) \colon v = 0 \text{ on } \Gamma_D^\iota\},$$

i.e. there exist constants $\alpha_0^\iota > 0$ such that

$$a^\iota(v,v) \geqslant \alpha_0^\iota\, \|v\|_{\boldsymbol{H}^1(\Omega^\iota)}^2 \text{ for every } v \in V^\iota.$$

Proof. From the ellipticity condition (2.2) on the coefficients a^{ι}_{ijkl} follows immediately that $a^{\iota}(v,v) \geqslant c^{\iota}_0 \int_{\Omega} \varepsilon_{ij}(v)\, \varepsilon_{ij}(v)$. Observe now that because of the Dirichlet condition, the only allowed rigid displacement in V^{ι} is $v=0$, so we have $V^{\iota} \cap \mathcal{R} = \{0\}$ and a direct application of corollary C.2, with $W = V^{\iota}$ finishes the proof. □

Theorem 3.10. *If the Γ^{ι}_D have positive measure, then the energy functional is coercive over V, and in particular over K. As a consequence the variational inequality*

$$a(u, v-u) \geqslant F(v-u) \textit{ for every } v \in K \tag{3.6}$$

has a unique solution $u \in K$.

Proof. This follows directly from lemma 3.9 and the continuity of F:

$$I^{\iota}(v) = \frac{1}{2} a^{\iota}(v,v) - F^{\iota}(v) \geqslant \alpha^{\iota}_0 \|v\|^2_{\boldsymbol{H}^1} - \|F^{\iota}\|_{\boldsymbol{H}^{-1}} \|v\|_{\boldsymbol{H}^1} \underset{\|v\| \to \infty}{\longrightarrow} \infty.$$

Thus the functional $I = I^a + I^b$ of the total energy is coercive and (3.6) has at least one solution by theorem A.5. Strict convexity of I would also ensure uniqueness, but we may just as well check it directly. To this end, let u, $w \in K$ be solutions. Then both $a(u, w-u) \geqslant F(w-u)$ and $a(w, u-w) \geqslant F(u-w)$ hold and adding them we immediately see by the non-negativity of $a(\cdot,\cdot)$ that

$$0 \leqslant -a(u,u) - a(w,w) \leqslant 0 \Longrightarrow u - w \in \mathcal{R}.$$

Recalling that $V \cap \mathcal{R} = \{0\}$ the proof is finished. □

The case meas $\boldsymbol{\Gamma^{\iota}_D = 0}$. This case may be solved in an analogous way to the previous one considering the quotients $W^{\iota} = \boldsymbol{H}^1(\Omega^{\iota}) / \mathcal{R}$, but this is not acceptable for contact problems because translations of one body with respect to the other are obviously relevant.[3.5] An alternative way is to introduce a compatibility condition on the data, as we do in the next theorem.

Theorem 3.11. *If* meas $\Gamma^{\iota}_D = 0$ *and the data f^{ι}, t^{ι} satisfy the **compatibility condition***

$$F^{\iota}(r) = \int_{\Omega^{\iota}} f^{\iota}_i r_i \,\mathrm{d}x + \int_{\Gamma^{\iota}_N} t^{\iota}_i r_i \,\mathrm{d}s_x < 0 \textit{ for every } r \in \mathcal{R}_2 := \{r \in K_0 \cap \mathcal{R}: -r \notin K_0\}, \tag{3.7}$$

where $K_0 = K - \tilde{g}$ is the cone resulting of translating the set K by some function $\tilde{g} \in K$ such that $\tilde{g}^a_{\nu} + \tilde{g}^b_{\nu} = g$, then (3.6) has a solution which is unique up to a member of

$$M = \{r \in \mathcal{R}: F(r) = 0\}.$$

Proof. [3.6] To prove existence notice first that clearly $V^{\iota} = \boldsymbol{H}^1(\Omega^{\iota})$ and consider the closed, convex and bounded sets:

$$K_{\rho} = \{v \in K: \|v\|_V \leqslant \rho\}.$$

Convexity and Gâteaux-differentiability of $I = I^a + I^b$ together with the boundedness of K_{ρ} allow us to apply proposition A.3, then theorem A.1 to deduce existence of solutions $u_{\rho} \in K_{\rho}$ to the problems

$$a(u_{\rho}, v - u_{\rho}) \geqslant F(v - u_{\rho}) \text{ for every } v \in K_{\rho}. \tag{3.8}$$

3.5. For this approach see for instance [10, Chapter III, §3].

3.6. "Our" proof is essentially a stripped-down version of [26, §3, §4, §5].

We now claim that *it is enough to prove that one of the* u_ρ *has norm strictly smaller than* ρ. For if u_ρ is such a solution, then for any $v \in K$ we have that the convex combination $w_\theta := u_\rho + \theta(v - u_\rho) \in K$ and for θ small enough: $\|w_\theta\| \leqslant \|u_\rho\| + \theta \|v - u_\rho\| < \rho$. Therefore $w_\theta \in K_\rho$ and we may substitute it for v in (3.8) obtaining

$$a(u_\rho, \theta(v - u_\rho)) \geqslant F(\theta(v - u_\rho)) \Rightarrow a(u_\rho, v - u_\rho) \geqslant F(v - u_\rho) \text{ for every } v \in K.$$

Let us assume then, to arrive at a contradiction, that for every $\rho > 0$ it is $\|u_\rho\| = \rho$, where we may as well assume $\rho > 1$. Now let us normalize u_ρ as $w_\rho = \rho^{-1} u_\rho$. Because of $0 \in K$ and convexity, we have $w_\rho \in K$. Now define $\mathcal{E}(v) = |\varepsilon(v)|^2$ as in (C.1), $\varepsilon(\cdot)$ being as before the linearized strain tensor, and use the ellipticity condition (2.2) on a_{ijkl} to write $a(v, v) \geqslant c\,\mathcal{E}(v)$ for some constant $c > 0$. Choosing $v = 0$, the inequality in (3.8) flips and using the continuity of F we find

$$\rho\,\|F\| \geqslant F(u_\rho) \geqslant a(u_\rho, u_\rho) \geqslant c\,\mathcal{E}(u_\rho) \tag{3.9}$$

and consequently $\mathcal{E}(w_\rho) = \rho^{-2}\,\mathcal{E}(u_\rho) \leqslant (\rho\,c)^{-1}\,\|F\|$, or:

$$\mathcal{E}(w_\rho) \longrightarrow 0 \text{ as } \rho \to \infty. \tag{3.10}$$

The boundedness of the w_ρ implies that a subsequence $w_{\rho_k} \rightharpoonup w_0 \in V$ weakly, and $\mathcal{E}$ being w.s.l.s.c. this means $\mathcal{E}(w_0) = 0$.[3.7] Also, because K is closed and convex $w_0 \in K$.

Define now $N := \ker\varepsilon \times \ker\varepsilon \subset V$ and observe that by proposition 3.4 the subspace N is finite dimensional, hence it is complete and the orthogonal projection $P: V \to N$ is well defined. We have the following property, a direct consequence of Korn's inequality:

$$c\,\mathcal{E}(v) \geqslant \inf_{r \in N} |v - r|^2 \text{ for every } v \in V. \tag{3.11}$$

Indeed, that infimum is by definition attained at Pv and $v - Pv \in N^\perp$, so we may apply corollary C.2 and $\mathcal{E}(Pv) = 0$ to deduce $|v - Pv|^2 \leqslant c\,\mathcal{E}(v - Pv) = c\,\mathcal{E}(v)$.

We claim next that *there exists* $c > 0$ *such that* $|Pw_\rho| \geqslant c > 0$ *uniformly on* ρ: for otherwise a subsequence $(w_{\rho_k}) \subset (w_\rho)$ would have the property $|Pw_{\rho_k}| \to 0$ and using (3.10) we would find

$$|w_{\rho_k}| \leqslant |w_{\rho_k} - Pw_{\rho_k}| + |Pw_{\rho_k}| \leqslant c\,\mathcal{E}(w_{\rho_k})^{1/2} + |Pw_{\rho_k}| \longrightarrow 0.$$

But this is a contradiction because we know by construction and Korn's inequality that

$$1 = \|w_{\rho_k}\|^2 \leqslant |w_{\rho_k}|^2 + \mathcal{E}(w_{\rho_k}).$$

Observe that in the finite-dimensional space N we have $Pw_\rho \to Pw_0 = w_0$ and hence $|w_0| = |Pw_0| > 0$ which implies $w_0 \neq 0$. But remember that $w_0 \in K \cap N$ and if we had $K \cap N = \{0\}$ we would already have a contradiction. In our case we proceed as follows. Given that $F(w_0) < 0$ and $Pw_\rho \to w_0$, there exists some $\beta > 0$ with

$$F(Pw_\rho) \leqslant \beta \tag{3.12}$$

for every $\rho > \rho_\beta$ and some suitable ρ_β. We also have

$$\begin{array}{rcll} c\,\mathcal{E}(u_\rho) & \leqslant & F(u_\rho - Pu_\rho) + F(Pu_\rho) & \text{(by (3.9))} \\ & \leqslant & \|F\|\,\|u_\rho - Pu_\rho\| + F(Pu_\rho) & \\ & \leqslant & c\left(|u_\rho - Pu_\rho| + \mathcal{E}(u_\rho - Pu_\rho)^{1/2}\right) + F(Pu_\rho) & \text{(Korn's ineq.)} \\ & \leqslant & c\left(\mathcal{E}(u_\rho)^{1/2} + \mathcal{E}(u_\rho - Pu_\rho)^{1/2}\right) + F(Pu_\rho) & \text{(by (3.11))} \\ & = & c\,\mathcal{E}(u_\rho)^{1/2} + F(Pu_\rho) & \text{(since } \mathcal{E}(Pu_\rho) = 0) \\ & \leqslant & c\sqrt{\rho} + F(Pu_\rho) & \end{array}$$

3.7. $\mathcal{E}$ is clearly convex and Gâteaux-differentiable by the same arguments as I^ι.

Subtracting now $F(Pu_\rho) = \rho\, F(Pw_\rho)$ at both sides of the resulting inequality and using (3.12) we get $\beta\,\rho \leqslant -\rho\, F(Pw_\rho) \leqslant c\,\sqrt{\rho}$ or

$$0 < \beta \leqslant c\,\rho^{-1/2}.$$

Since this must happen for all $\rho > \rho_\beta$ we have an impossibility that finishes the proof of existence.

Concerning uniqueness, it is obvious that adding any function r_0 in M to a solution of (3.6) results in another solution if the sum is in K: being in $\mathcal{R}$, r_0 does not alter the elastic potential and being in M the external forces are zero, so the solution is unaltered. Conversely, if $u,\ w \in K$ are solutions then both $a(u,\ w - u) \geqslant F(w - u)$ and $a(w,\ u - w) \geqslant F(u - w)$ hold. Adding these like in the proof of theorem 3.10 we have that $r := u - w \in \mathcal{R}$ and using corollary 3.5:

$$\begin{cases} 0 \;=\; a(u, -r) \;\geqslant\; F(-r), \\ 0 \;=\; a(w, r) \;\geqslant\; F(r), \end{cases}$$

hence $F(r) = 0$ and therefore $r \in M$. □

A word is in order with respect to the compatibility condition (3.7). Firstly, because $\mathcal{R} = \ker a(\cdot, \cdot)$ it is clear that $F(r) \leqslant 0$ must hold for all $r \in \mathcal{R}_2$. Indeed, using $0 \in K$ as test function we have on the one hand $a(u, -u) \geqslant F(-u)$ or $-a(u, u) + F(u) \geqslant 0$ and in the other hand, using $r \in \mathcal{R}_2$ as the test function we obtain $C = -a(u, u) + F(u) \geqslant F(r)$. But $\mathcal{R}_2$ is a cone which leads to the condition $C \geqslant F(\lambda\, r)$ for all $\lambda > 0$ and therefore $F(r) \leqslant 0$ as stated.

Secondly, instead of the set $\mathcal{R}_2 = \{r \in K_0 \cap \mathcal{R}\colon -r \notin K_0\}$ one could try simply $K \cap \mathcal{R}$, but this set is not a cone (to see this, take any translation approaching both bodies and multiply by an arbitrary constant $\lambda > 0$). However, if we require $-r \notin K_0$, then we know that $-r \cdot \nu > 0$ or $r \cdot \nu < 0$, that is: the rigid movement r, and any multiple thereof (hence $\mathcal{R}_2$ is a cone), increases the distance between the bodies. Consequently the compatibility condition ensures that any such rigid displacement increases the total energy of the deformation. Finally, uniqueness is guaranteed up to a rigid displacement which doesn't alter the work done by the external loads, i.e. $r \in \mathcal{R}_2$ such that $F(r) = 0$.

4. NUMERICAL ANALYSIS

For the implementation of a numerical approximation we chose a finite element method with a so-called *mortar method* for the discretization and an *active/inactive dual set strategy* for the resolution of the non-linearity stemming from the non-penetration condition.[4.1] The former was first introduced in the 1990s as a domain decomposition technique for non-matching grids enabling the "gluing" of different discretizations at the common interface,[4.2] and has since proved to have wide applicability in contact problems. The latter is an iterative scheme to reduce the non-linearity at the contact zone to a choice between Neumann and Dirichlet boundary conditions at each node.

We denote with subindices α and β nodes in the meshes for the bodies and with φ^β and ψ^α basis functions in suitable finite dimensional spaces $\hat{V}$ and $\hat{Q}'$ respectively (see below). The idea behind the (dual-space) mortar method is to replace the strong non-penetration condition by a weak form:

$$\int_{\hat{\Gamma}_S^a} u^\alpha \nu^\alpha \psi^\alpha \leqslant \int_{\hat{\Gamma}_S^a} g \psi^\alpha \tag{4.1}$$

for each node α in the discretization of the boundary Γ_S^a, then couple the nodal basis functions at the interface of the two bodies using a change of coordinates. This is is done by means of a *coupling matrix* M of entries

$$M_{\alpha\beta} := \int_{\hat{\Gamma}_S^a} \varphi^\beta \psi^\alpha.$$

Because we will be using the dual Lagrange basis for the coupling and contact condition the matrix M has a very simple structure and we may conveniently use (4.1).[4.3] The basis resulting after the transformation is non-conforming but the new system has the same structure as for the Signorini problem with one body. The strategy is then to iteratively resolve the non-linearity in the possible contact zone by solving at each step a new problem where a subset of the nodes in $\hat{\Gamma}_S^a$ is marked as being of Dirichlet type and the rest as Neumann. After every step the computed solution is used to decide to which of these subsets any given node in $\hat{\Gamma}_S^a$ belongs by means of formula (4.5).

4.1. Saddle point formulation.

In what follows one body is chosen to be the *master* or *mortar* and the other the *slave* or *non-mortar*. The choice is arbitrary but has to be fixed and we choose $\mathcal{B}^a$ to be the slave body. Once and for all we set $n=2$. Before we discretize the problem, we transform it into a saddle point one. We first introduce a Lagrange multiplier space $Q := \boldsymbol{H}_{00}^{-1/2}(\Gamma_S^a)$. In a manner analogous to lemma B.5 we may define both the normal and tangential components of any $\mu \in Q$ and define the set

$$Q_+ := \left\{ \mu \in Q \colon \mu_\tau = 0, \mu_\nu \leqslant 0 \text{ on } \Gamma_S^a \text{ in } H_{00}^{-1/2}(\Gamma_S^a) \right\},$$

4.1. Because a complete treatment of the required theoretical background is beyond the scope of this thesis, we refer to the book [15, Chapter 7] for an exposition of finite element techniques in variational inequalities, to [22, Chapters 4 and 6] for a thorough introduction to contact problems and to [17, Chapter 2] for a treatment of two-dimensional problems. In particular, we won't be addressing any of the fundamental issues of well-posedness, error estimation, stability or convergence.

4.2. C. Bernardi, Y.Maday and A. T. Patera: "A New Nonconforming Approach to Domain Decomposition: the Mortar Element Method", Collège de France seminar. Pitman (1990), pp. 13–51.

4.3. This idea of using dual Lagrange multipliers departs from the original mortar method and was first introduced in [33].

which can be seen to be a closed and convex cone. Let also $v_\nu^\iota = \gamma^0_{\Gamma_S^\iota,\nu}(v^\iota)$ as in proposition B.6 and for every $v \in V = V^a \times V^b$, $\mu \in Q$ define the bilinear form

$$b(v,\mu) := \langle \mu_\nu, v_\nu^a \rangle + \langle \mu_\nu, v_\nu^b \rangle$$

using for v_ν^b in the second term the projections onto the sets U with φ^a and φ^b (cf. §2.2.1 and figure 2.2). Then the *saddle point formulation* of the contact problem is:

Problem 4.1. *Find* $(u,\lambda) \in V \times Q_+$ *such that*

$$\begin{cases} a(u,v) + b(v,\lambda) &= F(v), \qquad v \in V, \\ b(u,\mu-\lambda) &\leqslant g((\mu-\lambda)_\nu), \quad \mu \in Q_+. \end{cases}$$

Theorem 4.1. *Problem 4.1 has a solution. If* $(u,\lambda) \in V \times Q_+$ *solves problem 4.1, then* u *solves problem 3.3 and* (u,λ) *is a saddle point for the Lagrange functional*

$$L(v,\mu) := I(v) + b(v,\mu) - g(\mu_\nu).$$

Proof. Existence is proved in [15, Theorem 7.11], once the necessary growth condition for b is verified. The other statements follow from [15, Lemma 7.9]. ☐

4.2. Discretization.

Given a triangulation with simplices or quadrilaterals of each of the bodies, we choose standard P1 or Q1 bilinear finite elements and denote φ^β the basis function associated to vertex β. The Lagrange multiplier space Q is discretized using bilinear dual basis functions ψ^α, which are again associated to the vertices and take the value two at their associated vertex α and minus one at the others in the element. These basis functions must fufill the following biorthogonality condition, and will therefore be different in a three dimensional setting:

$$\int_\Gamma \varphi^\beta \psi^\alpha = \delta_{\beta\alpha} \int_\Gamma \varphi^\beta \text{ for every } \alpha, \beta. \tag{4.2}$$

We denote by $\hat{V}$ and $\hat{Q}$ the resulting discretizations. A will be the stiffness matrix of entries

$$A_{ik}^{\alpha\beta} := \int_{\hat{\Omega}} a_{ijkl} \varphi^\beta_{,l} \varphi^\alpha_{,j}$$

and B the matrix with

$$B_{\alpha\beta} := \int_{\hat{\Gamma}_S^a} \varphi^\beta \psi^\alpha$$

(recall that $b(\cdot,\cdot)$ is defined on $V \times Q$ via the duality pairing of Q). Then the discrete formulation of the first equation of problem 4.1 is

$$A\hat{u} + B\hat{\lambda} = F.$$

Note that because of the choice of dual basis functions ψ^β and the fact that the integral is over just a subset of the boundary of Ω^a, the matrix is actually $B = (0,D)^\top$ with D block diagonal of entries $D_{\alpha\alpha} = \int_{\hat{\Gamma}_S^a} \varphi^\alpha \psi^\alpha \, \mathrm{Id}_2$. In order to write the system in block matrix form we let $\mathcal{M}$ stand for the set of indices of the *master contact boundary* $\hat{\Gamma}_S^b$, $\mathcal{S}$ for the set of indices of the *slave contact boundary* $\hat{\Gamma}_S^a$ and $\mathcal{N}$ for the rest of the indices in both meshes. Without the boundary conditions we have the following block structure in the system:

$$\begin{pmatrix} A_{\mathcal{NN}} & A_{\mathcal{NM}} & A_{\mathcal{NS}} & 0 \\ A_{\mathcal{MN}} & A_{\mathcal{MM}} & A_{\mathcal{MS}} & -M^\top \\ A_{\mathcal{SN}} & A_{\mathcal{SM}} & A_{\mathcal{SS}} & D \end{pmatrix} \begin{pmatrix} u_\mathcal{N} \\ u_\mathcal{M} \\ u_\mathcal{S} \\ \lambda_\mathcal{S} \end{pmatrix} = \begin{pmatrix} F_\mathcal{N} \\ F_\mathcal{M} \\ F_\mathcal{S} \end{pmatrix}. \tag{4.3}$$

We now turn to the boundary conditions. As announced, we replace the pointwise non-penetration condition by its weak (dual) equivalent

$$\int_{\hat{\Gamma}_S^a} u^\alpha \, \nu^\alpha \, \psi^\alpha \leqslant \int_{\hat{\Gamma}_S^a} g \, \psi^\alpha.$$

Define $\hat{g}^\alpha := \int_{\hat{\Gamma}_S^a} g \, \psi^\alpha$ and $\hat{u}_\nu^\alpha := \nu^{\alpha\top} D_{\alpha\alpha} \, \hat{u}^\alpha$, then the condition reads

$$\hat{u}_\nu^\alpha \leqslant \hat{g}^\alpha \text{ for all } \alpha \in \mathcal{S}.$$

Analogously we set $\hat{\lambda}_\nu^\alpha := \nu^{\alpha\top} D_{\alpha\alpha} \, \hat{\lambda}^\alpha$ and recalling that $\lambda = \sigma(u) \, \nu$ this is used to discretize the condition that normal stresses are compressive as

$$\hat{\lambda}_\nu^\alpha \leqslant 0.$$

Finally, if the tangential part of $\hat{\lambda}$ at α is $\hat{\lambda}_\tau^\alpha = \hat{\lambda}^\alpha - \left(\hat{\lambda}_\nu^\alpha \, \nu^\alpha\right) \nu^\alpha$ we can write the discrete version of problem 4.1:

Problem 4.2. *Find* $\left(\hat{u}, \hat{\lambda}\right) \in \hat{V} \times \hat{Q}$ *such that*

$$\begin{cases} A\,\hat{u} + B\,\hat{\lambda} &=& F, \\ \hat{u}_\nu^\alpha &\leqslant& \hat{g}^\alpha, \\ \hat{\lambda}_\nu^\alpha &\leqslant& 0, \\ \hat{\lambda}_\nu^\alpha \, (\hat{u}_\nu^\alpha - \hat{g}^\alpha) &=& 0, \\ \hat{\lambda}_\tau^\alpha &=& 0. \end{cases}$$

In order to solve this problem the following iterative scheme is used, where at each step k the nodes in $\mathcal{S}$ are classified into **active** $\mathcal{A}_k$ and **inactive** $\mathcal{I}_k$:

Algorithm 4.1

1. Initialize $\mathcal{A}_0 = \emptyset$ and $\mathcal{I}_0 = \mathcal{S}$.
2. Solve
$$\begin{cases} A\,\hat{u}^k + B\,\hat{\lambda}^k &=& F, & \\ \hat{u}_\nu^{k\alpha} &=& \hat{g}^\alpha & \alpha \in \mathcal{A}_k, \\ \hat{\lambda}_\nu^{k\alpha} &=& 0 & \alpha \in \mathcal{I}_k, \\ \hat{\lambda}_\tau^{k\alpha} &=& 0 & \alpha \in \mathcal{I}_k. \end{cases} \tag{4.4}$$
3. Set the new active set:
$$\mathcal{A}_{k+1} := \left\{\alpha \in \mathcal{S} \colon \hat{\lambda}_\nu^{k\alpha} - c(\hat{u}_\nu^{k\alpha} - \hat{g}^\alpha) < 0\right\} \tag{4.5}$$
with $c > 0$ a constant, and let $\mathcal{I}_{k+1} = \mathcal{S} \backslash \mathcal{A}_{k+1}$.
4. Repeat from step 2. until $\mathcal{A}_{k+1} = \mathcal{A}_k$.

To justify the idea behind the recipe (4.5), suppose that $\left(\hat{u}^k, \hat{\lambda}^k\right)$ solves (4.4) and consider some vertex $\alpha \in \mathcal{S}$:

- If $\hat{\lambda}_\nu^{k\alpha} - c(\hat{u}_\nu^{k\alpha} - \hat{g}^\alpha) < 0$ then: if $\alpha \in \mathcal{A}_k$ it must be $\hat{\lambda}_\nu^{k\alpha} < 0$, because of the second line of (4.4), in compliance with the requisite that there is normal and compressive stress whenever there is contact. If $\alpha \in \mathcal{I}_k$ then $\hat{\lambda}^{k\alpha} = 0$ by (4.4) and $\hat{u}_\nu^{k\alpha} - \hat{g}^\alpha > 0$ holds, which violates the non-penetration condition. The node is then marked as active forcing it to $\hat{u}_\nu^{k\alpha} = \hat{g}^\alpha$.

- On the other hand, if $\hat{\lambda}_\nu^{k\alpha} - c(\hat{u}_\nu^{k\alpha} - \hat{g}^\alpha) \geqslant 0$ then, either $\hat{\lambda}_\nu^{k\alpha} \geqslant 0$ because it was $\alpha \in \mathcal{A}_k$ or $\hat{u}_\nu^{k\alpha} - \hat{g}^\alpha < 0$ if it was $\alpha \in \mathcal{I}_k$. In the former case we move the node to $\mathcal{I}_k$ thus imposing $\hat{\lambda}_\nu^{k\alpha} = 0$.

Note that in the first situation an inactive node is set as active with a Dirichlet condition fixing it and in the second an active node is set as inactive with a Neumann condition.

To obtain the algebraic representation of problem (4.4) we must yet define a modified basis for $\hat{V}$ with the coordinate transformation given by

$$T = \begin{pmatrix} \mathrm{Id} & & \\ & \mathrm{Id} & D^{-1} M \\ & & \mathrm{Id} \end{pmatrix}$$

and the new system is $T A_k T^\top \hat{u}^k + T B_k T^\top \hat{\lambda}^k = T F_k$ (see [18, §5]). After including the boundary conditions and applying T we arrive at a linear system ([18, Eq. (5.4)]) at each step which we may solve with a direct method.

4.3. Test problem and implementation details.

For the implementation of the algorithm described we used the open source C++ framework DUNE [3, 2] in particular its modules DUNE-GRID and DUNE-ISTL [5] with SuperLU [25, 24, 8] as linear solver. The output was visualized and postprocessed with the open source tool PARAVIEW.[4.4]

For simplicity we considered simple geometries, two two-dimensional rectangular plates, one resting over the other, and solved the equations for isotropic materials with Lamé coefficients determined from Young's modulus and Poisson's ratio. We chose Q1 elements in uniform meshes, although given the facilities provided by DUNE, it poses no problem to use much more complicated meshes once any issues due to the realization of boundary conditions are taken into account (for instance, the constraints may impose a different choice for A_0, as explained in [18, §6.2]).

Among the many details of the implementation we'd like to point out the following items.

The computation of the integrals $\int_{\hat{\Gamma}_S^a} \varphi^\alpha \psi^\beta$ proves to be quite costly because of the need to intersect the supports. Thanks to the choice of dual Lagrange basis functions, locally supported on the vertices, this task is simplified (the geometries we considered allowed for substantial optimization in this respect, albeit *ad-hoc* and therefore of little real value). Another advantage of the dual Lagrange multipliers is that the variables may be eliminated from the linear system to be solved, resulting in yet fewer computations.

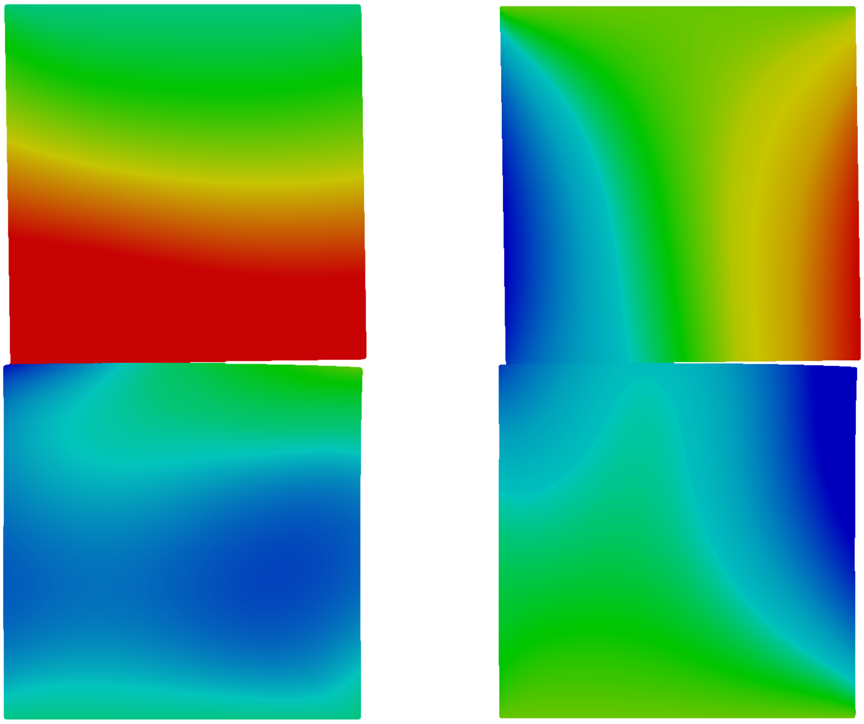
The determination of the gap function requires the projection of nodes of the master side onto the slave side along the normal field of the master boundary, again a computationally intensive task which for our simple geometry was straightforward. In more general situations this could be very costly.

Applying the coordinate transformation in a naive way to the system results in horrendous slowdown due to the cost of sparse matrix multiplication. It is however straightforward to use the block structure of the matrices to greatly reduce the number of operations, as suggested in [18].

Finally, due to time constraints, we left out many interesting questions: local refinement of the grid near the contact zone, experiments on the influence of its size and shape, other geometries, comparison with Hertz' formulas, experiments regarding the "small displacements" assumption (when does the model break?), visualization using von Mises stress, etc.

4.4. Mainly developed by KITWARE, INC. See www.paraview.org.

Test problem. Following the example given in [18], we modelled two homogeneous, isotropic square plates $\Omega^a = (0,1) \times (0,1)$ and $\Omega^b = (0,-1) \times (-1,0)$ with Young's modulus $E = 8 \times 10^9$ Pa (approximately "soft" wood) and Poisson's ratio $\nu = 0.3$.



(a) Horizontal displacements **(b)** Vertical displacements

Figure 4.1. Test problem.

For the boundary we set homogeneous Dirichlet conditions at the top of the slave body Ω^a and the bottom of the master Ω^b and Neumann conditions at the sides: pushing down and inwards at the left side of Ω^a with pressure $(3, -4) \times 10^6$ Pa and at the right side of Ω^b with pressure $(-3, -7) \times 10^6$ Pa. The two remaining sides were left with homogeneous Neumann conditions. The gap function is simply $g \equiv 0$ along the common segment $\{0\} \times [0, 1]$. Because the resulting deformations were very small (as they only could, since we work in the framework of linear elasticity), we applied a correcting factor of 10 to make them more apparent. The result is displayed in figures 4.1(a) and 4.1(b).

Appendix A. Convex minimization

In this section V will denote a reflexive real Banach space and K a non-empty closed convex set. We let $I: K \to \overline{\mathbb{R}}$ be a functional with range on $[-\infty, +\infty]$ and set to solve the problem

$$I(u) = \min_{v \in K} I(v),$$

as well as to characterize its solutions by diverse means as a function of the properties of K and I. In our context of solid mechanics the functional I represents the total energy of the body and the set K the admissible displacements, which include the essential boundary conditions to the classical problem and other mechanical restrictions. We need the following definitions:

1. I is **convex** whenever for every $u, v \in V$ and $\theta \in [0, 1]$:

$$I(\theta u + (1 - \theta) v) \leqslant \theta I(u) + (1 - \theta) I(v).$$

2. I is **proper** if $I \not\equiv +\infty$ and $I(x) > -\infty$ always.

3. I is **Gâteaux-differentiable** at $u \in V$ if there exists a linear continuous functional $DI(u) \in V'$ such that for every $v \in V$:

$$\lim_{\lambda \to 0} \frac{I(u + \lambda v) - I(u)}{\lambda} = \langle DI(u), v \rangle_{V' \times V}. \tag{A.1}$$

We note that the left-hand side is the directional derivative of I at u along the direction v and that for convex I the numerator is a monotone function of λ implying that the limit always exists in $\overline{\mathbb{R}}$. What matters however is that this limit admits the representation on the right-hand side.

4. I is **coercive** if

$$\lim_{\|u\| \to \infty} I(u) = +\infty.$$

5. I is **(weakly) sequentially lower semicontinuous** (**w.s.l.s.c.** resp. **s.l.s.c.**) if whenever a sequence $(v_n) \subset V$ converges (weakly) to a $v \in V$ the following holds

$$\liminf I(v_n) \geqslant I(v).$$

Note that the weak topology is not first countable and as such may not be *expected* to have sequences characterize its closed sets (even though this is not a necessary condition). As a matter of fact, one can prove that for any set $U \subset V$ one has $\operatorname{Cl} U \subseteq \operatorname{Cl}_{\text{weak}} U$ but the opposite inclusion is in general false.[A.1] This is precisely one of the interesting features of convex sets, where one has the equality of both closures.

The following theorem is the key to the existence result we need.

Theorem A.1. *(Generalized Weierstraß minimization theorem) Let K be a non-empty closed convex subset of a reflexive Banach space V and let $I: K \to R$ be a w.s.l.s.c. proper functional. If either K is bounded or I is coercive, then I attains its minimum over K: there exists at least one $u \in K$ such that*

$$I(u) = \min_{v \in K} I(v).$$

A.1. See e.g. Stephen Willard, "General topology", §8.9 for the details.

Proof. Assuming first that K is bounded, we prove that I must be bounded below on K. For if it were not, we could find a sequence $(v_m) \subset K$ such that $I(v_m) < -m$ for every $m \in \mathbb{N}$, and K being convex and bounded, a subsequence (v_{m_k}) would exist which converged weakly to some $v \in K$. But then, because I is w.s.l.s.c. we'd have $-\infty = \liminf I(v_{m_k}) \geqslant I(v)$, an impossibility because I is proper.

Because $\alpha := \inf \{I(v) \colon v \in K\} > -\infty$, we can choose a minimizing sequence $(v_m) \subset K$, such that $I(v_m) \searrow \alpha$ and it will be bounded. Then it will have a subsequence $(v_{m_k}) \subset K$ weakly converging to some $v \in K$, because K is weakly closed by convexity. Finally, I being w.s.l.s.c. we arrive at $\alpha = \liminf I(v_{m_k}) \geqslant I(v) = \alpha$.

Assume now that K is only closed and convex but I is coercive. Once more we check that I must be bounded below: coerciveness of I means that for every $M > 0$ there exists $R \geqslant 0$ such that $I(v) > M$ whenever $\|v\|_V \geqslant R$. Therefore I is bounded below in $\{\|v\| \geqslant R\} \cap K$ and by the previous reasoning, it is bounded below in the bounded and convex set $\{\|v\| < R\} \cap K$. We would now like to choose a minimizing sequence as before and extract a weakly convergent subsequence from it, but for this we first need to show that the sequence is bounded:

Let $M_0 > 0$ and $R_0 \geqslant 0$ such that $I(v) > M_0$ whenever $\|v\|_V \geqslant R_0$. Because I is bounded below, $m = \inf \{I(v) \colon v \in K, \|v\| \geqslant R_0\}$ is finite and we can again find $R \geqslant R_0$ such that $I(v) > m$ whenever $\|v\|_V \geqslant R$. Set now $\alpha := \inf \{I(v) \colon v \in K, \|v\| \leqslant R\}$. This is finite, in particular $\alpha \leqslant m$ and in fact it is a global infimum by construction. If $(v_m) \subset K$ is a minimizing sequence, it will eventually be contained in $\{\|v\| \leqslant R\} \cap K$, so it is bounded and contains a weakly convergent subsequence (v_{m_k}) which by convexity has a limit $v \in K$. Because I is w.s.l.s.c. we have as before $\alpha = \liminf I(v_{m_k}) \geqslant I(v) = \alpha$ and this concludes the proof. $\square$

The following characterization of convex functionals will be useful (e.g. in theorem 3.2 to prove that solutions of problem 3.1 are minimizers in problem 3.3).

Proposition A.2. *Let $I \colon K \to \mathbb{R}$ be Gâteaux-differentiable. Then I is convex if and only if for every $v, w \in K$: $I(v) - I(w) \geqslant \langle DI(w), v - w \rangle$.*

Proof. If I is convex, then for $\theta \in (0,1)$ and $v \neq w$ in K:

$$\begin{aligned} I(w + \theta (v - w)) - I(w) &\leqslant \theta I(v) + (1 - \theta) I(w) - I(w) \\ &= \theta I(v) - \theta I(w). \end{aligned}$$

Dividing by θ and taking the limit as $\theta \to 0^+$ we have

$$I(v) - I(w) \geqslant \lim_{\theta \to 0^+} \frac{1}{\theta} [I(w + \theta (v - w)) - I(w)] = \langle DI(w), v - w \rangle.$$

To prove the converse implication note first that

$$\langle DI(w), w - v \rangle \geqslant I(w) - I(v) \geqslant \langle DI(v), w - v \rangle$$

and this means that *the derivative is monotone*:

$$\langle DI(w) - DI(v), w - v \rangle \geqslant 0. \tag{A.2}$$

Define $h \colon [0,1] \to \mathbb{R}$ for any two $v, w \in K$ as

$$h(s) = I(s w + (1 - s) v) = I(v + s (w - v)).$$

We now check that this function is non-decreasing. First:

$$h'(s) = \langle DI(v + s (w - v)), w - v \rangle$$

and therefore

$$\begin{aligned}[h'(s)-h'(t)]\,(s-t) &= \langle DI(v+s\,(w-v))-DI(v+t\,(w-v)),(s-t)\,(w-v)\rangle\\ &= \langle DI(v+s\,(w-v))-DI(v+t\,(w-v)),\\ &\qquad\qquad v+s\,(w-v)-(v+t\,(w-v))\rangle\\ &\overset{(A.2)}{\geqslant} 0.\end{aligned}$$

So h is indeed non-decreasing on $[0,1]$ and we have

$$\frac{h(1)-h(s)}{1-s}\geqslant\frac{h(s)-h(0)}{s-0}\Longrightarrow h(s)\leqslant s\,h(1)+(1-s)\,h(0),$$

or, recalling the definition h:

$$I(s\,w+(1-s)\,v)\leqslant s\,I(w)+(1-s)\,I(v),$$

which is exactly the convexity of I. ☐

Because our set of admissible displacements K is not bounded, we see that the key properties to prove are weak lower semicontinuity and coercivity. To this end we have the next three results.

Proposition A.3. *Let $I: K\to\mathbb{R}$ be convex and Gâteaux-differentiable. Then I is w.s.l.s.c.*

Proof. Let $v_m\rightharpoonup v$ in $K\subset V$. Using proposition A.2 and the fact that $DI(u)\in V'$, we see:

$$\liminf_{m\to\infty}\,(I(v_m)-I(v))\geqslant\liminf_{m\to\infty}\,\langle DI(v),v_m-v\rangle_{V'\times V}=0.$$

But this means exactly that I is w.s.l.s.c. ☐

The following two theorems are key in the proofs of existence and equivalence of the formulations. We refer to [12, Chapter 1] and [29, Chapter 2] for more complete versions.

Proposition A.4. *Let $I: K\to\mathbb{R}$ be Gâteaux-differentiable on the closed convex $\emptyset\neq K\subset V$ and assume it attains a minimum at $u\in K$. Then:*

$$\langle DI(u),v-u\rangle\geqslant 0 \text{ for every } v\in K.$$

Proof. Let $u\in K$ be a minimizer for I and let $v\in K$, $v\neq u$, $\theta\in(0,1)$. Because K is convex, $u+\theta\,(v-u)\in K$ and because u is a minimizer $I(u)\leqslant I(u+\theta\,(v-u))$. But then:

$$0\leqslant\frac{1}{\theta}\,[I(u+\theta\,(v-u)-I(u))]\underset{\theta\to 0}{\longrightarrow}\langle DI(u),v-u\rangle\geqslant 0.$$

☐

Theorem A.5. *Let K be a non-empty closed convex subset of a reflexive Banach space V and let $I: K\to R$ be a convex, Gâteaux-differentiable and coercive functional. Then there exists at least one $u\in K$ where I attains its minimum.*

Proof. By proposition A.3 I is w.s.l.s.c. and because it is also coercive we may apply theorem A.1 finishing the proof. ☐

Appendix B. Trace spaces

As above, let Ω be of class $C^{1,1}$ and Γ_D open in $\Gamma := \partial\Omega$ and $\Sigma := \operatorname{Int}(\Gamma \backslash \Gamma_D)$.[B.1] Special care must be had in order to define constraints while dealing with the traces of functions in $H^1(\Omega)$. A difficulty arises for instance when, as in our case, the interface $\partial\Sigma \cap \partial\Gamma_D$ is not empty: in this situation a homogeneous Dirichlet boundary condition forces the trace in $H^{1/2}(\Sigma)$ of the solution to have zero value at the boundary $\partial\Sigma$, which means that not every constraint defined in Σ can be fulfilled. Furthermore, the data g on Σ must define the initial gap and the normal displacement of the solution should be smaller than g. First we must be able to define this normal displacement and second we must compare it with g. The purpose of this section is to deal with these issues.[B.2]

B.1. The Lions-Magenes space $H^{1/2}_{00}(\Sigma)$.

A first constraint for the admissible displacements is given by the Dirichlet condition on Γ_D, meaning that we must look for solutions in the set $V := \{v \in H^1(\Omega) : \gamma(v)_{|\Gamma_D} = 0\}$. Consequently we have to consider the trace operator

$$\gamma^0 : V \rightarrow H^{1/2}(\Gamma)$$

in order to deal with the boundary conditions. Because any non-zero constant function over Σ is in $H^{1/2}(\Sigma)$, and it cannot be extended by zero through Γ_D and still be the trace of some $v \in H^1(\Omega)$, the restriction of the trace operator

$$\gamma^0_\Sigma : V \rightarrow H^{1/2}(\Sigma) : v \mapsto \gamma^0(v)_{|\Sigma}$$

is **not** surjective on $H^{1/2}(\Sigma)$. Another way of saying this is that the extension by zero of $H^{1/2}(\Sigma)$ to $H^{1/2}(\Gamma)$ is not possible. Among other things, this poses a problem at the generalization of Green's formula which allows to define the normal component of the stress tensor at the boundary, as we will see in §B.4.

One solution is to consider functions decaying fast enough at the border $\partial\Sigma$: let $\rho(x) = \operatorname{dist}(x, \partial\Sigma)$ on Σ, then define

$$H^{1/2}_{00}(\Sigma) := \left\{ v \in H^{1/2}(\Sigma) : \rho^{-1/2}\, v \in L^2(\Sigma) \right\}$$

with the scalar product and corresponding (equivalent) norm

$$(u, v)_{H^{1/2}_{00}(\Sigma)} := (u, v)_{H^{1/2}(\Sigma)} + \left(\rho^{-1/2}\, u, \rho^{-1/2}\, v \right)_{L^2(\Sigma)}$$

$$\|u\|^2_{H^{1/2}_{00}(\Sigma)} = \|u\|^2_{H^{1/2}(\Sigma)} + \left\| \rho^{-1/2}\, u \right\|^2_{L^2(\Sigma)}.$$

The space so defined is a Hilbert space properly and continuously embedded in $H^{1/2}(\Sigma)$, whose definition is actually independent of ρ, as long as we take some positive function decaying to zero as the distance to the boundary. Our main interest in this space originates in the following theorem.

B.1. For simplicity of notation we will work throughout this section as if there were just one body.

B.2. We refer to [14, §1.3] for detailed definitions and properties of trace spaces $H^s(\Gamma)$ in general and integration in them.

Theorem B.1. *The trace operator γ_Σ^0 is surjective onto $H_{00}^{1/2}(\Sigma)$ for smooth $\partial\Sigma$.*

Proof. See [14, Theorem 1.5.3.4]. □

Besides the preceding result there are many alternative ways of seeing this space, all intricately related:

- It is the interpolation space $(H_0^1(\Sigma), L^2(\Sigma))_{1/2}$. See [27, §11.7], [32, §33].
- It is the space of functions $\left\{ f \in H^{1/2}(\Sigma) \colon \tilde{f} \in H^{1/2}(\Gamma) \right\}$ where $\tilde{f}$ is the extension by zero to all of Γ. See [14, §1.3-§1.5].
- It is the space of functions $\left\{ f \in H_0^{1/2}(\Sigma) \colon \rho^{-1/2} f \in L^2(\Sigma) \right\}$ where ρ is the distance to the boundary $\partial\Sigma$. See [14, Corollary 1.4.4.10].
- It is the domain of $-\Delta^{1/4}$, Δ being the *Friedrichs extension* to $L^2(\Sigma)$ of the Laplace operator defined on $C_0^\infty(\Sigma)$.[B.3] See [31].

B.2. Ordering in $H_{00}^{1/2}(\Sigma)$.

The kinematical non-penetration constraint is given as an inequality for smooth functions in §2.2.1. In order to make sense of this condition for functions in more general spaces we now recall the framework for partial orderings in normed spaces.[B.4]

As usual V will denote a reflexive Banach space. A subset $\mathcal{C} \subset V$ is a **cone** whenever $\lambda\, \mathcal{C} \subset \mathcal{C}$ for every $\lambda > 0$. We say that the **vertex** of the cone is at the origin if this holds for $\lambda \geqslant 0$ and that the vertex is at u_0 if the cone $\mathcal{C} - u_0$ has its vertex at the origin. A cone is called **proper** if it is convex and has vertex at the origin. When such a cone exists one can define the relation $u \mathcal{R} v \equiv u - v \in \mathcal{C}$ and prove that it is reflexive, transitive and antisymmetric and that as such it is a **partial order** on V. Because we are interested in orders that preserve the structure of the space we consider those with the properties

1. $u \geqslant v \Rightarrow \lambda\, u \geqslant \lambda\, v$ for every $\lambda > 0$
2. $u \geqslant v \Rightarrow u + w \geqslant v + w$ for every $w \in V$.

It can be proved that there exists a one-to-one correspondence between these partial orders preserving linear structure and the closed proper cones of V: having some structure preserving partial order defined we define the **positive cone** of V as the set

$$\mathcal{C}_+ := \{v \in V \colon v \geqslant 0\},$$

which will be closed and proper. Reciprocally, defining some such proper cone ensures the existence of a corresponding partial order as seen above.

A positive cone may be used to define a corresponding **dual positive cone**

$$\mathcal{C}_+^* := \{f \in V' \colon \langle f, v \rangle \geqslant 0 \text{ for all } v \in \mathcal{C}_+\}$$

which is always closed by the continuity of the duality pairing. Furthermore, in the case that $\mathcal{C}_+$ is closed, we may characterize its elements as

$$\mathcal{C}_+ = \{v \in V \colon \langle f, v \rangle \geqslant 0 \text{ for every } f \in \mathcal{C}_+^*\}.$$

B.3. The Friedrichs extension of a non-negative and densely defined symmetric operator is a canonical extension to a self adjoint operator.

B.4. For a complete discussion of these questions, see e.g [7, Chapter 19].

Let $\Gamma_S \subset \Sigma$.[B.5] We say that a function $v \in H_{00}^{1/2}(\Sigma)$ is **non-negative on $\boldsymbol{\Gamma_S}$ in (the sense of) $\boldsymbol{H_{00}^{1/2}(\Sigma)}$** and we write $v \geqslant 0$, whenever there exists a sequence $(v_m) \subset C^{0,1}(\Sigma)$ such that $v_m(x) \geqslant 0$ on Γ_S and $v_m \rightharpoonup v$ weakly in $H_{00}^{1/2}(\Sigma)$. In a similar way we may carry this definition to the open sets $U \subset \mathbb{R}^{n-1}$ used above and thus say that $v \in H_{00}^{1/2}(U)$ is **non-negative on $\boldsymbol{U'}$ in $\boldsymbol{H_{00}^{1/2}(U)}$** if some $(v_m) \subset C^{0,1}(U)$ exists such that $v_m \geqslant 0$ on U' and $v_m \rightharpoonup v$ in $H_{00}^{1/2}(U)$.

Proposition B.2. *The set*

$$\mathcal{C}_+ := \left\{ v \in H_{00}^{1/2}(\Sigma) \colon v \text{ is non-negative on } \Gamma_S \text{ in } H_{00}^{1/2}(\Sigma) \right\}$$

is a closed and convex cone in $H_{00}^{1/2}(\Sigma)$ with vertex at the origin. As such it induces a partial ordering in $H_{00}^{1/2}(\Sigma)$ which is compatible with its vector structure and we denote with $\geqslant$. We have

$$u \geqslant v \text{ in } H_{00}^{1/2}(\Sigma) \Leftrightarrow u - v \in \mathcal{C}_+,$$

which justifies the notation $u \geqslant 0$ for non-negative functions on Γ_S in $H_{00}^{1/2}(\Sigma)$.

Proof. To check that $\mathcal{C}_+$ is closed, let $(v_m) \subset \mathcal{C}_+$ converge to some $v \in H_{00}^{1/2}(\Sigma)$. Each v_m being in $\mathcal{C}_+$, for every m there exists a sequence $\left(v_m^k\right)_{k \geqslant 0} \subset C^{0,1}(\Sigma)$ with $v_m^k \geqslant 0$ on Σ and $v_m^k \rightharpoonup v_m$ in $H_{00}^{1/2}(\Sigma)$. By Mazur's lemma, for each of these there exists a sequence of convex combinations of v_m^k, that is $\tilde{v}_m^k = \sum_{j=1}^k \lambda_j v_m^j$ with $\sum_j \lambda_j = 1$ and $\lambda_j \geqslant 0$, such that $\tilde{v}_m^k \to v_m$ in $H_{00}^{1/2}(\Sigma)$. For every m choose k_m such that $\left\| \tilde{v}_m^{k_m} - v_m \right\| < 1/m$ and build a sequence. Then let $\varepsilon > 0$ and choose m_0 such that $\|v_m - v\| < \varepsilon$ for all $m > m_0$ to find that $\left\| \tilde{v}_m^{k_m} - v \right\| \leqslant \left\| \tilde{v}_m^{k_m} - v_m \right\| + \|v_m - v\| < m^{-1} + \varepsilon$. The sequence $\tilde{v}_m^{k_m}$ converges strongly to v, hence weakly, is in $\mathcal{C}^{0,1}$ and $\tilde{v}_m^{k_m}(x) \geqslant 0$ on Γ_S. Therefore $\mathcal{C}_+$ is closed.

That $\mathcal{C}_+$ is a convex cone with vertex at the origin follows directly from the linearity of the weak convergence: if $v_m \rightharpoonup v$ and $\lambda \geqslant 0$, then $\lambda v_m \rightharpoonup \lambda v$ and $\lambda v \in \mathcal{C}_+$, and so on.

Finally the proof that the order induced is compatible with the linear structure is again a direct consequence of the linearity of the convergence in the definition. □

Using $\mathcal{C}_+^*$, the dual cone of $\mathcal{C}_+$, we have a definition for positivity in the dual space. We say that $w \in H_{00}^{-1/2}(\Sigma)$ is **non-negative on $\boldsymbol{\Gamma_S}$ in the sense of $\boldsymbol{H_{00}^{-1/2}(\Sigma)}$** and write $w \geqslant 0$, whenever $\langle w, v \rangle_{H_{00}^{-1/2}(\Sigma)} \geqslant 0$ for every $v \in H_{00}^{1/2}(\Sigma)$ which is non-negative on Γ_S in the sense of $H_{00}^{1/2}(\Sigma)$.

B.3. Normal and tangential components of the trace.

Here we study the existence of the normal and tangential components of the trace of a function, but before we need a simple lemma stating the "local character" of $\boldsymbol{W}^{s,p}(\Omega)$.

Lemma B.3. *Let $v \in \boldsymbol{W}^{s,p}(\Omega)$, $s > 0$ and let $h \in C^{\kappa,\lambda}(\overline{\Omega})$ with $\kappa \in \mathbb{N}_0, \lambda \in (0,1]$ and $\kappa + \lambda \geqslant s$ if $s \in \mathbb{N}$ or $\kappa + \lambda > s$ if $s \notin \mathbb{N}$. The product $h\, v$ is in $\boldsymbol{W}^{s,p}(\Omega)$ and there exists a constant $C = C(h, p, \Omega)$ such that*

$$\|h\, v\|_{\boldsymbol{W}^{s,p}(\Omega)} \leqslant c\, \|v\|_{\boldsymbol{W}^{s,p}(\Omega)}.$$

B.5. We already mentioned (cf. footnote 3.2 in page 11) that in the case that $\Gamma_S \Subset \Sigma$, the trace $\gamma_{\Gamma_S}^0$ is surjective onto $H^{1/2}(\Gamma_S)$ and then we could use the space $H^{1/2}(\Gamma_S)$ right away.

Proof. The case $s \in \mathbb{N}$, $\kappa = s$ follows easily from the boundedness of all derivatives of the function h. The case $s \in \mathbb{N}, \kappa = s - 1, \lambda = 1$ is also easy. We focus on the case $\kappa = 0$, $1 < s < \lambda \leqslant 1$, since the general case $s > 0, s \notin \mathbb{N}$ follows from it and the preceding ones.

On the one hand h is bounded in $\overline{\Omega}$ and we have trivially: $\|h\, u\|_p \leqslant \|h\|_\infty \|u\|_p$. On the other, for the Gagliardo seminorm

$$[\![h\, u]\!]_{s,p}^p := \iint_{\Omega \times \Omega} \frac{|h(x)\, u(x) - h(y)\, u(y)|^p}{|x - y|^{n+sp}} \mathrm{d}x\, \mathrm{d}y,$$

we use the convexity of $x \mapsto x^p$ twice (that is, we use $|a + b|^p \leqslant 2^{p-1}(|a|^p + |b|^p)$) and obtain

$$\begin{aligned} |h(x)\, u(x) - h(y)\, u(y)|^p &\leqslant C\, (|h(x)\, u(x) - h(x)\, u(y)|^p + |h(x)\, u(y) - h(x)\, u(x)|^p \\ &\qquad + |h(x)\, u(x) - h(y)\, u(x)|^p + |h(y)\, u(x) - h(y)\, u(y)|^p) \\ &= C(|h(x)|^p\, |u(x) - u(y)|^p + |h(x)|^p\, |u(y) - u(x)|^p \\ &\qquad + |h(x) - h(y)|^p\, |u(x)|^p + |h(y)|^p\, |u(x) - u(y)|^p). \end{aligned}$$

We plug this into the integral but using first that for the third summand:

$$|h(x) - h(y)|^p \leqslant C\, |x - y|^{\lambda p}.$$

Then:

$$\begin{aligned} [\![h\, u]\!]_{s,p}^p &\leqslant C\left(\|h\|_\infty^p\, 3 \iint_{\Omega \times \Omega} \frac{|u(x) - u(y)|^p}{|x - y|^{n+sp}} \mathrm{d}x\, \mathrm{d}y + \iint_{\Omega \times \Omega} \frac{C\, |x - y|^{\lambda p}\, |u(x)|^p}{|x - y|^{n+sp}} \mathrm{d}x\, \mathrm{d}y \right) \\ &\leqslant C\left([\![u]\!]_{s,p}^p + \iint_{\Omega \times \Omega} \frac{|u(x)|^p}{|x - y|^{n+(s-\lambda)p}} \mathrm{d}x\, \mathrm{d}y \right). \end{aligned}$$

Notice that the exponent in the denominator of the integrand is $n - \delta < n$ for some $\delta > 0$ so the function $1 / |z|^{n+(s-\lambda)p}$ is integrable in any bounded domain. We use Fubini-Tonelli and change the variable in the $\mathrm{d}y$ integral with $z = x - y$ (x is fixed) to obtain:

$$\begin{aligned} [\![h\, u]\!]_{s,p}^p &\leqslant C\left([\![u]\!]_{s,p}^p + \int_\Omega |u(x)|^p \int_{x-\Omega} \frac{1}{|z|^{n-\delta}} \mathrm{d}z\, \mathrm{d}x \right) \\ &\leqslant C\left([\![u]\!]_{s,p}^p + \int_\Omega |u(x)|^p \int_{\Omega+\Omega} \frac{1}{|z|^{n-\delta}} \mathrm{d}z\, \mathrm{d}x \right) \\ &\leqslant C\left([\![u]\!]_{s,p}^p + \|u\|_{\boldsymbol{L}^p(\Omega)} \right), \end{aligned}$$

where the second inequality is due to the fact that $x - \Omega \subset \Omega + \Omega$ for every $x \in \Omega$. Therefore

$$\|h\, u\|_{\boldsymbol{W}^{s,p}(\Omega)} \leqslant C\, \|u\|_{\boldsymbol{W}^{s,p}(\Omega)},$$

and keeping track of the constants above we see that indeed $C = C(h, p, \Omega)$. □

Proposition B.4. *Let Ω be bounded and with $C^{\kappa,1}$ boundary Γ, for $\kappa = 0, 1, \ldots$ Then the unit normal vector ν to Γ is well defined and in $C^{\kappa-1,1}$ for $\kappa > 0$, or L^∞ for $\kappa = 0$.*

Proof. First we recall that because of Rademacher's theorem,[B.6] any Lipschitz function is a.e. differentiable with L^∞ derivative. If $\varphi^i \colon U_i \to \Gamma$ is a $C^{\kappa,1}$ function whose epigraph contains $\Omega_i \subset \Omega$ and whose graph is $\Gamma_i \subset \partial\Omega$, we may define the outward pointing normal as

$$\nu(x', \varphi^i(x')) := \frac{(\nabla\, \varphi^i, -1)}{\|(\nabla\, \varphi^i, -1)\|},$$

B.6. See for instance [13].

and because $\varphi^i \in C^{\kappa,1}$, this is a $C^{\kappa-1,1}$ function if $\kappa > 1$ or L^∞ if $\kappa = 0$. ☐

With proposition B.4 we have for Ω of class $C^{1,1}$ that $\nu \in C^{0,1}$ and we can properly define for any $v \in \boldsymbol{H}^{1/2}(\Gamma)$ the decomposition

$$v = v_\nu \nu + v_\tau \text{ where } v_\nu = v \cdot \nu \text{ and } v_\tau = v - v_\nu \nu,$$

and thanks to lemma B.3 (applied to the function ν and some lifting $\tilde{v} \in \boldsymbol{H}^1(\Omega)$ of v) we know that $v_\nu \in H^{1/2}(\Gamma)$ and $v_\nu \nu \in \boldsymbol{H}^{1/2}(\Gamma)$.[B.7] Furthermore, we have:

Lemma B.5. *Let $\Gamma = \partial\Omega$ or an open subset thereof. The map $v \overset{\phi}{\mapsto} (v_\nu, v_\tau)$ is an isometric isomorphism*

$$\boldsymbol{H}^{1/2}(\Gamma) \simeq H^{1/2}(\Gamma) \oplus \boldsymbol{H}_\tau^{1/2}(\Gamma),$$

where

$$\boldsymbol{H}_\tau^{1/2}(\Gamma) := \left\{ v \in \boldsymbol{H}^{1/2}(\Gamma) \colon v_\nu = 0 \right\}$$

is a closed subspace of $\boldsymbol{H}^{1/2}(\Gamma)$ and the scalar product in $H^{1/2}(\Gamma) \oplus \boldsymbol{H}_\tau^{1/2}(\Gamma)$ is defined component-wise.

Proof. Let $\phi = (\phi_\nu, \phi_\tau) := (v \cdot \nu, v - (v \cdot \nu)\, \nu)$. The map is well defined because $v \cdot \nu \in H^{1/2}(\Gamma)$ and $v - \phi_\nu(v)\, \nu \in \boldsymbol{H}^{1/2}(\Gamma)$ by lemma B.3. It is clearly linear and it is also injective because if $\phi_\nu(v) = 0$ and $\phi_\tau(v) = 0$, then $v = \phi_\tau(v) + \phi_\nu(v)\, \nu = 0$. That it is surjective is clear once we take $w \in H^{1/2}(\Gamma)$ and $t \in \boldsymbol{H}^{1/2}(\Gamma)$ and let $v := w\, \nu + t \in \boldsymbol{H}^{1/2}(\Gamma)$. Furthermore it is an isometry since if $v \in \boldsymbol{H}^{1/2}(\Gamma)$, then (observe that the scalar product in $\boldsymbol{H}_\tau^{1/2}(\Gamma)$ is the one inherited from $\boldsymbol{H}^{1/2}(\Gamma)$):

$$\begin{aligned}
&(\phi_\nu(v), \phi_\nu(v))_{H^{1/2}} + (\phi_\tau(v), \phi_\tau(v))_{\boldsymbol{H}^{1/2}} \\
&\quad = (v \cdot \nu, v \cdot \nu)_{H^{1/2}} + (v - (v \cdot \nu)\, \nu, v - (v \cdot \nu)\, \nu)_{\boldsymbol{H}^{1/2}} \\
&\quad = (v \cdot \nu, v \cdot \nu)_{H^{1/2}} + (v, v)_{\boldsymbol{H}^{1/2}} + ((v \cdot \nu)\, \nu, (v \cdot \nu)\, \nu)_{\boldsymbol{H}^{1/2}} - 2\, (v, (v \cdot \nu)\, \nu)_{\boldsymbol{H}^{1/2}} \\
&\quad = (v_i \nu_i, v_j \nu_j)_{H^{1/2}} + (v, v)_{\boldsymbol{H}^{1/2}} + (v_j \nu_j \nu_i, v_j \nu_j \nu_i)_{H^{1/2}} - 2\, (v_i, v_j \nu_j \nu_i)_{H^{1/2}} \\
&\quad = (v, v)_{\boldsymbol{H}^{1/2}}.
\end{aligned}$$

Finally, by the open mapping theorem, the inverse of ϕ is also continuous and consequently ϕ is an isomorphism. ☐

Combining this lemma with theorem B.1 we have:

Proposition B.6. *The trace operator $\gamma_\Sigma^0 \colon V \twoheadrightarrow \boldsymbol{H}_{00}^{1/2}(\Sigma)$ is surjective and may be split into two surjective operators*

$$\gamma_{\Sigma_\nu}^0 \colon V \longrightarrow\!\!\!\!\!\rightarrow H_{00}^{1/2}(\Sigma) \text{ and } \gamma_{\Sigma_\tau}^0 \colon V \longrightarrow\!\!\!\!\!\rightarrow \boldsymbol{H}_{00,\tau}^{1/2}(\Sigma),$$

where

$$\boldsymbol{H}_{00,\tau}^{1/2}(\Sigma) := \left\{ v \in \boldsymbol{H}_{00}^{1/2}(\Sigma) \colon \gamma_{\Sigma_\nu}^0(v) = 0 \right\}$$

which coincide with the normal and tangential component of vectors along Σ, i.e. for $\psi \in C^1(\overline{\Omega})$ we have

$$\gamma_{\Sigma_\nu}^0(\psi) = \psi_{|\Sigma} \cdot \nu \text{ and } \gamma_{\Sigma_\tau}^0(\psi) = \psi_{|\Sigma} - (\psi_{|\Sigma} \cdot \nu)\, \nu.$$

B.7. Actually, we could require no more than a boundary of class $C^{1,1/2+\varepsilon}$ but there would obviously be little advantage to it.

Proof. It only remains to prove that the trace operators agree with the usual normal and tangent for C^1 functions, but this is a consequence of their being extensions of operators defined for smooth functions. □

The following immediate corollary will be necessary in the proof of theorem 3.1.

Corollary B.7. *Let $\varphi \in C^{\infty}(\overline{\Omega})$ be such that $\varphi \cdot \nu \leqslant 0$ on Σ and $\operatorname{supp} \varphi \subset \Sigma$, where $\nu \in C^{0,1}$ is the unit outward normal to Σ. If $F \in H_{00}^{-1/2}(\Sigma)$ satisfies*

$$\langle F, \gamma_{\Sigma_\nu}^0(\varphi) \rangle \geqslant 0$$

for all such φ, then $F \leqslant 0$ in $H_{00}^{-1/2}(\Sigma)$.

B.4. Integration by parts.

In the proof of theorem 3.1 it will be key to be able to integrate by parts to obtain the boundary conditions from the variational formulation. We will use an abstract form of Green's theorem due to Jean Pierre Aubin (slightly simplified), whose proof we take from [22, Theorem 5.8] and which will have as immediate consequences the particular cases that we need for the stress tensors.[B.8] We remark that an essential fact in theorem B.8 and the arguments of density allowing to deduce facts for functionals on trace spaces is the surjectivity of the trace operator which guarantees the existence of a right inverse for it. This is the main reason behind the use of the spaces $H_{00}^{1/2}$. In our particular problem, it will also be key to be able to split the trace into normal and tangential components which of course will be seen to coincide with the usual definitions in the smooth sense. This we do in corollary B.10.

Theorem B.8. *Let $V \subseteq \boldsymbol{H}^1(\Omega)$ and $S \subseteq L^2(\Omega)^{n \times n}$. Suppose the trace operator $\gamma: V \rightarrow Z$ is surjective and has kernel V_0. Let $A: V \rightarrow S$ be a continuous operator and $A_0^*: S' \rightarrow V'$ be the "formal adjoint" of its restriction to V_0, meaning that*

$$\langle A_0^* \sigma, v \rangle_{V' \times V} = \langle \sigma, A v \rangle_{S' \times S} \text{ for all } \sigma \in S' \text{ and all } v \in V_0. \tag{B.1}$$

Finally let $S_0' = \{\sigma \in S' : A_0^* \sigma \in \boldsymbol{L}^2(\Omega)\}$ and define a scalar product in it via $(\sigma, \tau)_{S_0'} := (\sigma, \tau)_{S'} + (A_0^* \sigma, A_0^* \tau)_{\boldsymbol{L}^2}$. There exists a uniquely determined linear and continuous operator $\pi: S_0' \rightarrow Z'$ such that

$$\langle \sigma, A v \rangle_{S' \times S} - \langle A_0^* \sigma, v \rangle_{V' \times V} = \langle \pi \sigma, \gamma v \rangle_{Z' \times Z} \text{ for all } \sigma \in S_0' \text{ and all } v \in V. \tag{B.2}$$

Proof. Let $\delta: Z \rightarrow V$ be a (continuous) right inverse of γ. Then $\delta \gamma: V \rightarrow V$ is a projection, has $\ker(\delta \gamma) = V_0$ and its transpose map $\gamma^* \delta^*: V' \rightarrow V'$ is a projection onto the subspace $V_0^{\perp} = \ker(\delta \gamma)^{\perp} = \{v^* \in V' : \langle v^*, v \rangle_{V' \times V} = 0 \text{ for all } v \in \ker(\delta \gamma)\}$.

If $j: V \hookrightarrow \boldsymbol{L}^2$ denotes the canonical embedding and j^* its adjoint, then $j^* A_0^*(S_0') \subset V'$. Let $\sigma \in S_0'$ and $v \in V_0$, then $A_0^* \sigma \in \boldsymbol{L}^2$ and if $A^*: S' \rightarrow V'$ is the adjoint of A

$$\begin{aligned} \langle (A^* - j^* A_0^*) \sigma, v \rangle_{V' \times V} &= \langle A^* \sigma, v \rangle_{V' \times V} - \langle j^* A_0^* \sigma, v \rangle_{V' \times V} \\ &= \langle A^* \sigma, v \rangle_{V' \times V} - (A_0^* \sigma, j v)_{\boldsymbol{L}^2} \\ &= \langle A^* \sigma, v \rangle_{V' \times V} - \langle A_0^* \sigma, v \rangle_{V' \times V} \\ &= 0. \end{aligned}$$

B.8. We also refer to the papers by Hühnlich and Naumann [19, 20] for a thorough treatment of boundary conditions in linear elasticity in a general framework.

Therefore $(A^* - j^* A_0^*)\,\sigma$ is in $V_0^\perp$, where we know that $\gamma^* \delta^*$ is the identity. If we define

$$\pi := \delta^* (A^* - j^* A_0^*) \colon S_0' \longrightarrow Z',$$

then we have

$$\gamma^* \pi = \gamma^* \delta^* (A^* - j^* A_0^*) = A^* - j^* A_0^*,$$

and for any $\sigma \in S_0'$ and $v \in V$ (note that (B.1) holds only in V_0):

$$\begin{aligned}\langle \sigma, A v\rangle_{S'\times S} - \langle A_0^* \sigma, v\rangle_{V'\times V} &= \langle A^* \sigma, v\rangle_{V'\times V} - \langle A_0^* \sigma, v\rangle_{V'\times V}\\ &= \langle (A^* - j^* A_0^*)\,\sigma, v\rangle_{V'\times V}\\ &= \langle \gamma^* \pi \sigma, v\rangle_{V'\times V}\\ &= \langle \pi\sigma, \gamma v\rangle_{Z'\times Z},\end{aligned}$$

as stated. One sees immediately that any choice of right inverse leads to this result, so we must only check that the operator fulfilling (B.2) is unique. To this purpose suppose there are two, π_1 and π_2, then using (B.2) we see that $\langle (\pi_1 - \pi_2)\,\sigma, \gamma v\rangle_{Z'\times Z} = 0$ for all $v \in V$, but the operator γ is surjective and thus $\langle (\pi_1 - \pi_2)\,\sigma, z\rangle_{Z'\times Z} = 0$ for every $z \in Z$ and this concludes the proof. □

This theorem allows us to generalize the concept of normal component at the boundary to a more abstract setting. Consider in particular the following case:

Corollary B.9. *In theorem B.8, set*

$$\begin{aligned} V &= \boldsymbol{H}^1(\Omega),\\ Z &= \boldsymbol{H}^{1/2}(\Gamma),\\ S &= \{\sigma \in L^2(\Omega)^{n\times n} \ \textit{symmetric}\}\\ A v &:= \varepsilon(v) = \frac{1}{2}(D v + D^\top v),\\ A_0^* \sigma &:= -\operatorname{div}\sigma. \end{aligned}$$

We have

$$S_0' \simeq \boldsymbol{H}^1_{\text{sym}}(\operatorname{div}) := \{\sigma \in L^2(\Omega)^{n\times n} \ \textit{symmetric} : \operatorname{div}\sigma \in \boldsymbol{L}^2(\Omega)\}$$

and

$$\int_\Omega \sigma \colon \varepsilon(v) + \int_\Omega \operatorname{div}(\sigma)\cdot v = \langle \pi(\sigma), \gamma(v)\rangle_{Z'\times Z}, \tag{B.3}$$

where we used the fact that $\operatorname{div}\sigma \in \boldsymbol{L}^2(\Omega)$ *and Riesz's representation theorem to write the second duality pairing in (B.2) with the scalar product of* $\boldsymbol{L}^2$. *Furthermore, if* $\sigma \in \boldsymbol{C}(\overline{\Omega})$, Ω *is* $C^{1,1}$ *then the outward pointing normal* ν *is well defined and Lipschitz and we have*

$$\pi(\sigma)_i = \sigma_{ij}\nu_j \ \textit{on}\ \Gamma.$$

Proof. Surjectivity of $\gamma \colon \boldsymbol{H}^1(\Omega) \twoheadrightarrow \boldsymbol{H}^{1/2}(\Gamma)$ is well known (see for instance [9]) and $V_0 := \ker\gamma = \boldsymbol{H}_0^1(\Omega)$, so proving (B.1) is a matter of using the density of $\boldsymbol{C}_0^\infty(\Omega)$ in this space and standard integration by parts. Hence formula (B.3) is a direct application of theorem B.8 and all that remains to be proved is the agreement of $\pi(\sigma)_i$ with $\sigma_{ij}\nu_j$ for smooth σ. First note that the existence of the outward normal as an element of $C^{0,1}$ is guaranteed by proposition B.4. We then apply the formula of integration by parts for smooth functions and equal it with that from theorem B.8 to find that

$$\int_\Gamma \sigma_{ij}\nu_j \cdot \psi_i = \langle \pi(\sigma), \psi_{|\Gamma}\rangle_{\boldsymbol{H}^{-1/2}(\Gamma)}$$

for every $\psi \in C^\infty(\overline{\Omega})$ and this finishes the proof. $\square$

Because of the last statement in the previous corollary, we *define*

$$\sigma_{ij}\nu_j := \pi(\sigma)_i \text{ for every } \sigma \in S_0',$$

where π is the operator associated to the trace and spaces from the corollary.

There are many possible choices in theorem B.8: for example we will be using below $V = \{v \in \boldsymbol{H}^1(\Omega) : \gamma_D(v) = 0\}$ and $Z = \boldsymbol{H}_{00}^{1/2}(\Sigma)$ with $\Sigma = \operatorname{Int}\Gamma \backslash \Gamma_D$ and then $\pi(\sigma)_i = \sigma_{ij}\nu_j$ on Σ, but *notice that Z depends on V if we want $\pi(\sigma)$ to agree with $\sigma_{ij}\nu_j$ for smooth σ.* Take for instance the trace $\gamma: V = \boldsymbol{H}^1(\Omega) \twoheadrightarrow Z = \boldsymbol{H}_{00}^{1/2}(\Gamma_N)$, which is surjective by theorem B.1. If we apply the formula of integration by parts for smooth functions and equal it with that from theorem B.8, we find for $\sigma \in \boldsymbol{C}^1(\Omega)$

$$\int_\Sigma \sigma\nu \cdot v = \langle \pi(\sigma), v_{|\Gamma_N} \rangle_{\boldsymbol{H}_{00}^{-1/2}(\Gamma_N)} \text{ for every } v \in \boldsymbol{C}^1(\Gamma),$$

but this tells us nothing about $\sigma\nu$ in $\Sigma \backslash \Gamma_N$.

Corollary B.10. *Let $\Sigma := \operatorname{Int}\Gamma \backslash \Gamma_D$. In theorem B.8, set:*

$$\begin{aligned} V &= \{v \in \boldsymbol{H}^1(\Omega) : \gamma_D(v) = 0\}, \\ Z &= \boldsymbol{H}_{00}^{1/2}(\Sigma), \\ S &= \{\sigma \in L^2(\Omega)^{n\times n} \text{ symmetric}\} \\ Av &= \varepsilon(v) = \frac{1}{2}(Dv + D^\top v), \\ A_0^*\sigma &= -\operatorname{div}\sigma. \end{aligned}$$

The trace operator $\gamma_\Sigma^0: V \twoheadrightarrow \boldsymbol{H}_{00}^{1/2}(\Sigma)$ is surjective and the integration by parts formula holds, with $S_0' \simeq \boldsymbol{H}_{\text{sym}}^1(\operatorname{div}) := \{\sigma \in L^2(\Omega)^{n\times n} \text{ symmetric} : \operatorname{div}\sigma \in \boldsymbol{L}^2(\Omega)\}$ and the operator π split into its normal and tangential components*:*

$$\pi_\nu: S_0' \longrightarrow H_{00}^{-1/2}(\Sigma) \text{ and } \pi_\tau: S_0' \longrightarrow \boldsymbol{H}_{00,\tau}^{-1/2}(\Sigma).$$

That is, we have the following formula:

$$\int_\Omega \sigma : \varepsilon(v) + \int_\Omega \operatorname{div}(\sigma) \cdot v = \langle \pi_\nu(\sigma), \gamma_{\Sigma_\nu}^0(v) \rangle_{H_{00}^{-1/2}(\Sigma)} + \langle \pi_\tau(\sigma), \gamma_{\Sigma_\tau}^0(v) \rangle_{\boldsymbol{H}_{00,\tau}^{-1/2}(\Sigma)},$$

where the trace operators are those given in proposition B.6. Furthermore, if $\sigma \in \boldsymbol{C}^1(\overline{\Omega})$ then

$$\pi(\sigma)_i = \sigma_{ij}\nu_j, \quad \pi_\nu(\sigma) = \sigma_{ij}\nu_j\nu_i \text{ and } \pi_\tau(\sigma)_i = \sigma_{ij}\nu_j - \pi_\nu(\sigma)\nu_i.$$

For general $\sigma \in \boldsymbol{H}_{\text{sym}}^1(\operatorname{div})$, we define $\sigma_\nu := \pi_\nu(\sigma)$ and $\sigma_\tau := \pi_\tau(\sigma)$.

Proof. As before the hypothesis of theorem B.8 are easily checked (theorem B.1). The applications π_ν and π_τ are defined using π from theorem B.8, the isomorphism from lemma B.5 (valid for $\boldsymbol{H}_{00}^{1/2}(\Sigma)$ as well) and the corresponding Riesz maps $\zeta: \boldsymbol{H}_{00}^{1/2}(\Sigma) \leftrightarrow \boldsymbol{H}_{00}^{-1/2}(\Sigma)$ $\zeta_\nu: H_{00}^{1/2}(\Sigma) \leftrightarrow H_{00}^{-1/2}(\Sigma)$ and $\zeta_\tau: \boldsymbol{H}_{00,\tau}^{1/2}(\Sigma) \leftrightarrow \boldsymbol{H}_{00,\tau}^{-1/2}(\Sigma)$ (recall that $\boldsymbol{H}_{00,\tau}^{1/2}(\Sigma)$ is a closed subspace) as follows:

$$\pi_\nu := \zeta_\nu^{-1} \circ \phi_\nu \circ \zeta \circ \pi \text{ and } \pi_\tau := \zeta_\tau^{-1} \circ \phi_\tau \circ \zeta \circ \pi.$$

The last statement follows in the same manner as in the proof of corollary B.9. $\square$

Appendix C. Korn's inequality

As usual we suppose $\Omega \Subset \mathbb{R}^n$ open and with Lipschitz (that is, $C^{0,1}$) boundary. We set

$$\mathcal{E}(v) := \int_\Omega \varepsilon_{ij}(v)\varepsilon_{ij}(v)\,\mathrm{d}x \quad \text{and} \quad |v|^2 := \int_\Omega v_i v_i\,\mathrm{d}x, \tag{C.1}$$

then Korn's inequality states that $\mathcal{E}(v) + |v|^2 \geqslant c\,\|v\|^2_{\boldsymbol{H}^1(\Omega)}$ for all $v \in \boldsymbol{H}^1(\Omega)$ and is the central piece in the proof of coercivity of the energy functional in the preceding sections. Note that this is not a triviality because on the left-hand side there are more terms with combined partial derivatives including some that could take negative values, whereas all the terms on the right-hand side are non-negative. What is perhaps more remarkable is the fact that there's no scalar equivalent in the following sense. Let $v \in L^2(\Omega)$ be a scalar function. In order to have $v \in H^1(\Omega)$, the sum of the squares of all first partial derivatives must be finite, that is: if we define $q(\xi) := \xi_i \xi_i$ for $\xi \in \mathbb{R}^n$, then

$$Q(v) := \int_\Omega q\,(\nabla\, v)\,\mathrm{d}x = \|\nabla\, v\|_{\boldsymbol{L}^2(\Omega)} \text{ must be finite.}$$

Alternatively one could try to use another quadratic form for this, *but positive definiteness is necessary:* if $q(\xi) = 0$ for some $\xi \neq 0$, then the function $v(x) = \xi \cdot x$ has constant gradient $\nabla\, v = \xi$ and one has $Q(v) = 0$. However $\|\nabla\, v\|_{\boldsymbol{L}^2(\Omega)} = |\xi|\,|\Omega| > Q(v)$. As soon as Q has non trivial kernel an inequality like (C.1) fails for scalar functions. However the form $\mathcal{E}$ in (C.1) has a non-trivial kernel (of dimension $n\,(n-1)\,/\,2$ by proposition 3.4) and yet the inequality holds.

In simpler elliptic problems like Poisson's equation we may use the Poincaré-Friedrichs inequality in order to prove coercivity, and its applicability depends on some sort of essential constraint: Dirichlet boundary conditions on a subset of the boundary with positive measure, or functions with zero average for example. We will encounter a similar phenomenon in corollary C.2, where we must exclude functions in the kernel of $\mathcal{E}$.

The proof of theorem C.1 we studied relies on a celebrated lemma by Jacques Louis Lions stating regularity of distributions with weak derivatives in $H^{-1}(\Omega)$.[C.1] We will use a generalization found in the paper [1, §2] by Amrouche and Girault, which we state in lemma C.3 below.

Theorem C.1. ***Korn's inequality.*** *There exists a constant $c > 0$ such that*

$$\mathcal{E}(v) + |v|^2 \geqslant c\,\|v\|^2_{\boldsymbol{H}^1(\Omega)} \textit{ for every } v \in \boldsymbol{H}^1(\Omega). \tag{C.2}$$

Proof. Let $E = \{v \in \boldsymbol{L}^2(\Omega) \colon \varepsilon_{ij}(v) \in L^2(\Omega)\}$. We will prove that $E = \boldsymbol{H}^1(\Omega)$, even though this seems *a priori* not the case given that the fact that the sums $v_{i,j} + v_{j,i}$ are in $L^2(\Omega)$ does not necessarily imply that each of the terms is. With the scalar product

$$(u, v)_E := \int_\Omega \varepsilon_{ij}(u)\varepsilon_{ij}(v)\,\mathrm{d}x + \int_\Omega u_i v_i\,\mathrm{d}x$$

the space E is Hilbert. To see that it is indeed complete take a Cauchy sequence $(v^m) \subset E$: then all (v_i^m) and $\varepsilon_{ij}(v^m)$ are Cauchy in $L^2(\Omega)$, therefore convergent to limits v_i and w_{ij} in $L^2(\Omega)$. Because $L^2(\Omega) \subset L^1_{\mathrm{loc}}(\Omega)$, $\varepsilon_{ij}(v)$, with $v = (v_1, \ldots, v_n)$, and w_{ij} define distributions in $\mathcal{D}'(\Omega)$ and if we show that they are equal we will have $v \in E$. To show this take $f \in \mathcal{D}(\Omega)$. By definition

$$2\,(\varepsilon_{ij}(v), \varphi)_{\mathcal{D}'} := -(v_i, \varphi_{,j})_{\mathcal{D}'} - (v_j, \varphi_{,i})_{\mathcal{D}'},$$

C.1. See [10, §3.3], in particular theorem 3.2.

and these distributions being regular we have

$$\begin{aligned}-\lim_{m\to\infty}\left(\int_\Omega v_i^m\,\varphi_{,j}\,\mathrm{d}x+\int_\Omega v_j^m\,\varphi_{,i}\,\mathrm{d}x\right) &= \lim_{m\to\infty}\left(\int_\Omega v_{i,j}^m\,\varphi\,\mathrm{d}x+\int_\Omega v_{j,i}^m\,\varphi\,\mathrm{d}x\right)\\ &= \lim_{m\to\infty}\int_\Omega 2\varepsilon_{ij}(v^m)\varphi\,\mathrm{d}x\\ &= 2\int_\Omega w_{ij}\,\varphi\,\mathrm{d}x,\end{aligned}$$

where we used that for bounded Ω one has $\|u^m-u\|_{L^1}\leqslant\|u^m-u\|_{L^p}\to 0\Rightarrow\int u^m\to\int u$.

We now prove that the Hilbert space E is actually $\boldsymbol{H}^1(\Omega)$. On the one hand, the inclusion $\boldsymbol{H}^1(\Omega)\subseteq E$ is immediate; on the other, we take any $v\in E$ and want to apply equality (C.3) of lemma C.3 to the distributions $v_{i,j}$. Simply cancelling terms we arrive at the representation

$$v_{i,jk}=\varepsilon_{ik,j}(v)+\varepsilon_{ij,k}(v)-\varepsilon_{jk,i}(v).$$

Because $\varepsilon_{ij}(v)\in L^2(\Omega)$ we have $\varepsilon_{ij,k}(v)\in H^{-1}(\Omega)$ and consequently $v_{i,jk}\in H^{-1}(\Omega)$ as well. We now may apply (C.3) to the $v_{i,j}$ to conclude that in fact $v_{i,j}\in L^2(\Omega)$, that is: $v\in\boldsymbol{H}^1(\Omega)$.

Finally, the embedding j: $\boldsymbol{H}^1(\Omega)\hookrightarrow E$ is a continuous bijection and by the open mapping theorem j^{-1} is continuous: $\|v\|_E\geqslant\|j^{-1}\|\,\|v\|_{\boldsymbol{H}^1(\Omega)}$ and this concludes the proof. □

The consequence of Korn's inequality that we shall be using is the following:

Corollary C.2. *Let $W\subset\boldsymbol{H}^1(\Omega)$ be a closed subspace such that $W\cap\mathcal{R}=\{0\}$, $\mathcal{R}$ being the set of rigid displacements. Then the following inequalities hold*

$$\mathcal{E}(v)\geqslant c\,|v|^2 \text{ for every } v\in W \text{ and some } c>0$$

and

$$\mathcal{E}(v)\geqslant c\,\|v\|^2_{\boldsymbol{H}^1(\Omega)} \text{ for every } v\in W \text{ and some (other) } c>0.$$

Proof. Notice first that the second statement follows immediately from the first and Korn's inequality:

$$\|v\|^2_{\boldsymbol{H}^1(\Omega)}\leqslant c\,(\mathcal{E}(v)+|v|^2)\leqslant\tilde{c}\,\mathcal{E}(v).$$

Now, in order to prove the first inequality, we may consider just the case $|v|=1$, because for arbitrary v we can write

$$1\,c\leqslant\mathcal{E}(v\,|v|^{-1})=|v|^{-2}\,\mathcal{E}(v)\Rightarrow\mathcal{E}(v)\geqslant c\,|v|^2.$$

Suppose then, to arrive at a contradiction, that there is no such constant for functions $v\in W$ with $|v|=1$: we can find a sequence of scalars $c_m\to 0$ and a sequence of functions $(v^m)\subset W$ with $|v^m|=1$ such that $\mathcal{E}(v^m)\leqslant c_m\to 0$. By Korn's inequality, $c\,\|v^m\|^2_{\boldsymbol{H}^1}\leqslant c_m+1$, i.e. the v^m are bounded in $\boldsymbol{H}^1$. Eberlein's theorem implies then that a subsequence (v^{m_k}) converges weakly to some v^0 in the convex W and convexity and Gâteaux differentiability of $\mathcal{E}$ imply by proposition A.3 that $\mathcal{E}$ is w.s.l.s.c., that is:

$$\mathcal{E}(v^0)\leqslant\liminf\mathcal{E}(v^{m_k})=0.$$

Thus $\mathcal{E}(v^0)=0$ but we know from corollary 3.5 that $\mathcal{E}$ only vanishes over the rigid movements $\mathcal{R}$ and the only function $v\in W\cap\mathcal{R}$ is $v^0=0$. Consequently $v^m\rightharpoonup 0$ in $\boldsymbol{H}^1$.

Finally, the compact embedding j: $\boldsymbol{H}^1(\Omega^\iota)\hookrightarrow\boldsymbol{L}^2(\Omega^\iota)$ makes this weakly convergent sequence into one strongly convergent in $\boldsymbol{L}^2$. This is $v^{m_k}\to 0$ in $\boldsymbol{L}^2$, a contradiction to the assumption $|v^m|=1$ and the corollary is proved. □

Lemma C.3. *(Generalized Lion's lemma). Let* $\Omega\subset\mathbb{R}^n$ *be a bounded open set with Lipschitz boundary and let* $m\in\mathbb{Z}$, $p\in(1,\infty)$ *and*

$$X_{m,p}(\Omega):=\{v\in\mathcal{D}'(\Omega): \nabla v\in\boldsymbol{W}^{m-1,p}(\Omega)\}.$$

Then

$$X_{m,p}(\Omega)=W^{m,p}(\Omega).$$

And in particular

$$X_{0,2}(\Omega)=L^2(\Omega). \tag{C.3}$$

Proof. See [1, §2]. □

Appendix D. Notation and conventions

Throughout this work we use the *summation convention* where sums are carried over repeated indices in their obvious ranges. For brevity, we use the superscript $\iota = a,\ b$ whenever a symbol, property, etc. applies to both bodies. We will say that a property holds *locally* in $\Omega' \subset \Omega$ if (rather imprecisely), around each $x \in \Omega'$ there exists an open set where it is true.

The following table summarizes most of the notation used.

$A \Subset B$	$\overline{A} \subset \mathring{B}$ and $\overline{A}$ is compact.
$\gamma: V \twoheadrightarrow Z$	γ is surjective onto Z.
ι	Index for the bodies. $\iota \in \{a, b\}$
i, j	Indices for coordinates. Range $1, \ldots, n$
α, β	Indices for vertices in the discretization of the bodies.
$\mathcal{B}^\iota$	The body ι.
Ω^ι	Bounded open subset of $\mathbb{R}^n$ representing the reference configuration of the body $\mathcal{B}^\iota$.
$\Gamma_S^\iota, \Gamma_D^\iota, \Gamma_N^\iota$	Subsets of the boundary of Ω where boundary conditions of Signorini, Dirichlet and Neumann type respectively are given.
$x = (x', x_n)$	Decomposition of a vector x in its first $n-1$ coordinates $x' = (x_1, \ldots, x_{n-1})$ and x_n
$u^\iota = (u^{\iota\prime}, u_n^\iota)$	Decomposition of a deformation u^ι in its first $n-1$ coordinates $u^{\iota\prime} = (u_1^\iota, \ldots, u_{n-1}^\iota)$ and u_n^ι.
$u = (u^a, u^b)$	Deformation of the bodies.
$\nu^\iota(x)$	Unit outward normal vector to Γ^ι at x.
$w \cdot v = w_i v_i$	Scalar product in $\mathbb{R}^n$.
$u_\nu^\iota := u^\iota \cdot \nu^\iota$	Normal displacement at the boundary of the body ι.
$\sigma_\nu^\iota := \sigma_{ij}^\iota \nu_j^\iota \nu_i^\iota$	Normal tension at the boundary of the body ι.
$C^{k,\lambda}$	The class of k times differentiable, Hölder continuous functions with exponent λ, i.e. satisfying $\|D^\omega f(x) - D^\omega f(y)\| \leqslant c\, \|x - y\|^\lambda$ for every multi-index ω with $\lvert\omega\rvert \leqslant k$.
$L^p, W^{m,p}, H^s$	The usual spaces of *real valued* functions.
$\boldsymbol{L}^p, \boldsymbol{W}^{m,p}, \boldsymbol{H}^s$	The spaces of *vector valued* functions $(L^p)^n$, $(W^{m,p})^n$ and $(H^s)^n$.
$H_{00}^{1/2}(\Sigma)$	The space of functions in $H^{1/2}(\Sigma)$ "rapidly decaying" to zero on $\partial\Sigma$ (see §B.1).
$(\cdot,\cdot)$, $(\cdot,\cdot)_V$	Scalar product. The subindex is added where there may be ambiguity.
$\langle\cdot,\cdot\rangle$, $\langle\cdot,\cdot\rangle_{V'\times V}$	Duality pairing of a space V and its dual V'.
$\lvert\cdot\rvert$	L^2 or $\boldsymbol{L}^2$ norm.
$[f]$	Distribution in $\mathcal{D}'(\Omega)$ induced by some $f \in L_{\mathrm{loc}}^1(\Omega)$.

Bibliography

[1] Chérif Amrouche and Vivette Girault. Decomposition of vector spaces and applications to the Stokes problem in arbitrary dimension. *Czechoslovak Mathematical Journal*, 44(1), 1994.

[2] Peter Bastian, Markus Blatt, Andreas Dedner, Christian Engwer, Robert Klöfkorn, Ralf Kornhuber, Mario Ohlberger and Oliver Sander. A generic grid interface for parallel and adaptive scientific computing. Part II: implementation and tests in DUNE. *Computing*, 82(2):121–138, 2008.

[3] Peter Bastian, Markus Blatt, Andreas Dedner, Christian Engwer, Robert Klöfkorn, Mario Ohlberger and Oliver Sander. A generic grid interface for parallel and adaptive scientific computing. Part I: abstract framework. *Computing*, 82(2):103–119, 2008.

[4] Faker Ben Belgacem, P Hild and P Laborde. The mortar finite element method for contact problems. *Mathematical and Computer Modelling*, 28(4–8):263–271, oct 1998.

[5] Markus Blatt and Peter Bastian. The iterative solver template library. In *Proceedings of the 8th international conference on Applied parallel computing: state of the art in scientific computing*, pages 666–675. Berlin, Heidelberg, 2007. Springer-Verlag.

[6] Paolo Boieri, Fabio Gastaldi and David Kinderlehrer. Existence, uniqueness, and regularity results for the two-body contact problem. *Applied Mathematics & Optimization*, 15:251–277, 1987.

[7] Claudio Baiocchi and António Capelo. *Variational and quasivariational inequalities: applications to free boundary problems*. John Wiley & Sons, 1984.

[8] James W Demmel, Stanley C Eisenstat, John R Gilbert, Xiaoye S Li and Joseph W H Liu. A supernodal approach to sparse partial pivoting. *SIAM J. Matrix Analysis and Applications*, 20(3):720–755, 1999.

[9] Françoise Demengel and Gilbert Demengel. *Functional spaces for the theory of elliptic partial differential equations*, volume 8 of *Universitext*. Springer, 2012.

[10] Georges Duvaut and Jacques Louis Lions. *Inequalities in mechanics and physics*, volume 219 of *Grundlehren der mathematischen Wissenschaften*. Springer-Verlag, 1976.

[11] Christof Eck, Harald Garcke and Peter Knabner. *Mathematische Modellierung*. Springer-Lehrbuch. Springer, Stuttgart, Regensburg, Erlangen, 2. edition, 2011.

[12] Ivar Ekeland and Roger Temam. *Convex Analysis and Variational Problems*. Society for Industrial and Applied Mathematics, 1999.

[13] Lawrence Craig Evans and Ronald F. Gariepy. *Measure Theory and Fine Properties of Functions*. CRC Press, dec 1991.

[14] Pierre Grisvard. *Elliptic Problems in Nonsmooth Domains*. Classics in Applied Mathematics. Society for Industrial and Applied Mathematics, 2011.

[15] Christian Grossmann, Hans-Görg Roos and Martin Stynes. *Numerical Treatment of Partial Differential Equations*. Universitext. Springer Berlin Heidelberg, Berlin, Heidelberg, 2007.

[16] Heinrich Hertz. Über die Berührung fester elastischer Körper. *Journal für die reine und angewandte Mathematik*, 92:156–171, 1881.

[17] I. Hlaváček, J. Haslinger, J. Nečas and J. Lovšek. *Solution of variational inequalities in mechanics*, volume 66 of *Applied Mathematical Sciences*. Springer-Verlag, 1st edition, 1988.

[18] S Hüeber and Barbara I Wohlmuth. A primal–dual active set strategy for non-linear multibody contact problems. *Computer Methods in Applied Mechanics and Engineering*, 194(27-29):3147–3166, jul 2005.

[19] Rolf Hünlich and Joachim Naumann. On general boundary value problems and duality in linear elasticity. I. *Aplikace matematiky*, 23(3):208–230, 1978.

[20] Rolf Hünlich and Joachim Naumann. On general boundary value problems and duality in linear elasticity. II. *Aplikace matematiky*, 25(1):11–32, 1980.

[21] J J Kalker. Variational Principles of Contact Elastostatics. *IMA Journal of Applied Mathematics*, 20(2):199–219, 1977.

[22] Noboru Kikuchi and John Tinsley Oden. *Contact problems in elasticity. A study of variational inequalities and finite element methods.* SIAM Studies in Applied and Numerical Mathematics. Society for Industrial and Applied Mathematics, Philadelphia, 1. edition, 1988.

[23] David Kinderlehrer and Guido Stampacchia. *An introduction to variational inequalities and their applications*, volume 31 of *SIAM's classics in applied mathematics*. Society for Industrial and Applied Mathematics, Philadelphia, 2000.

[24] Xiaoye S Li. An overview of SuperLU: Algorithms, implementation, and user interface. *ACM Transactions on Mathematical Software*, 31(3):302–325, 2005.

[25] Xiaoye S Li, James W Demmel, John R Gilbert, Laura Grigori, Meiyue Shao and Ichitaro Yamazaki. SuperLU Users' Guide. Technical Report, sep 1999.

[26] Jacques Louis Lions and Guido Stampacchia. Variational inequalities. *Communications on pure and applied mathematics*, 20(3):493–519, 1967.

[27] Jacques Louis Lions and Enrico Magenes. *Non-homogeneous boundary value problems and applications, I*, volume 181 of *Die Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen*. Springer, 1972.

[28] Jerrold E. Marsden and Thomas J. R. Hughes. *Mathematical foundations of elasticity*. Dover civil and mechanical engineering, California Institute of Technology, Pasadena, Reprint of the 1983 edition edition, 1994.

[29] John Tinsley Oden. *Qualitative methods in nonlinear mechanics*. Prentice-Hall, New Jersey, 1986.

[30] M J Puttock and E G Thwaite. Elastic compression of spheres and cylinders at point and line contact. Technical Report 25, Melbourne, 1969.

[31] Norikazu Saito and Hiroshi Fujita. Remarks on traces of H^1-functions defined in a domain with corners. *Journal of Mathematical Sciences, the University of Tokio*, 7(2):325–345, 2000.

[32] Luc Tartar. *An Introduction to Sobolev Spaces and Interpolation Spaces*, volume 3 of *Lecture Notes of the Unione Matematica Italiana*. Springer, jan 2007.

[33] Barbara I Wohlmuth. A Mortar Finite Element Method Using Dual Spaces for the Lagrange Multiplier. *SIAM Journal on Numerical Analysis*, 38(3):989–1012, jan 2001.